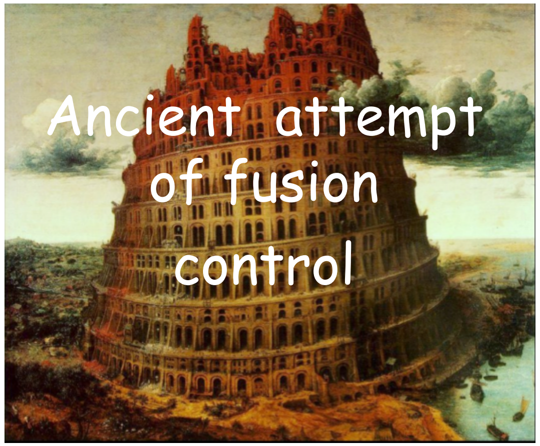
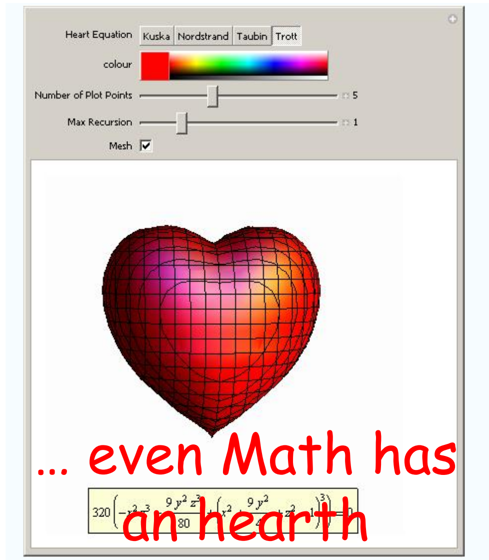
Analytical Solution of the Grad Shafranov equation in an elliptical prolate Geometry

F. Crisanti

ENEA-Frascati-Italy





Grad-Shafranov Equation

Let 's to restart from the Equilibrium equation

$$-\Delta^{*}\psi = \mu_0 R^2 P'(\psi) + II'(\psi) = \mu_0 R J_\varphi$$

The operator Δ* has the same structure (but a sign) of a normal Laplacian operator

$$\Delta^{*}\psi = \frac{\partial^2\psi}{\partial R^2} - \frac{1}{\mathrm{R}}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial z^2} = \mu_0 R J_\varphi$$

$$\Delta f = \frac{\partial^2 f}{\partial R^2} + \frac{1}{\mathrm{R}}\frac{\partial f}{\partial R} + \frac{\partial^2 f}{\partial z^2}$$

Laplace Equation

The Laplace equation $\Delta\Phi=0$, for a given system of coordinates is said "separable" or "*R* separable" when it admits solutions of the type

$$\Phi(u_1,u_2,\ldots u_n) = U^1(u_1)U^2(u_2)\ldots U^n(u_n)$$

$$\Phi(u_1,u_2,\ldots u_n) = \frac{U^1(u_1)U^2(u_2)\ldots U^n(u_n)}{R(u_1,u_2,\ldots u_n)}$$

In the case of rotational coordinates (like the toroidal one) the possibly admitted solution is the "*R* separable"

A necessary condition to get this is that

$$\frac{\sqrt{g}}{g_i} = R^2 f_i(u_1)F(u_1,u_2,\ldots u_n)$$

Where g_i are the metric elements

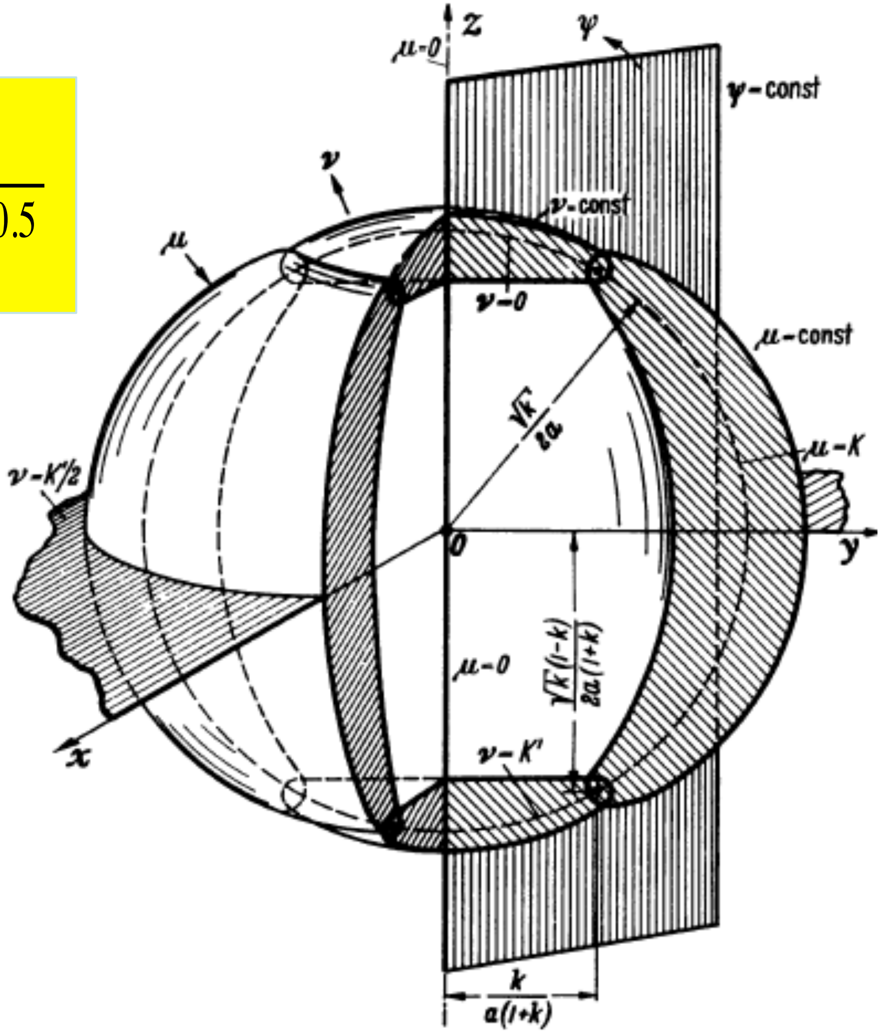
Let me to introduce the Cap-Cyclides Coordinates System

$$R + iZ = \frac{1}{a_s \; sn(\mu + i\nu) + i a_s k^{-0.5}} + \frac{i}{2 a_s k^{-0.5}}$$

With

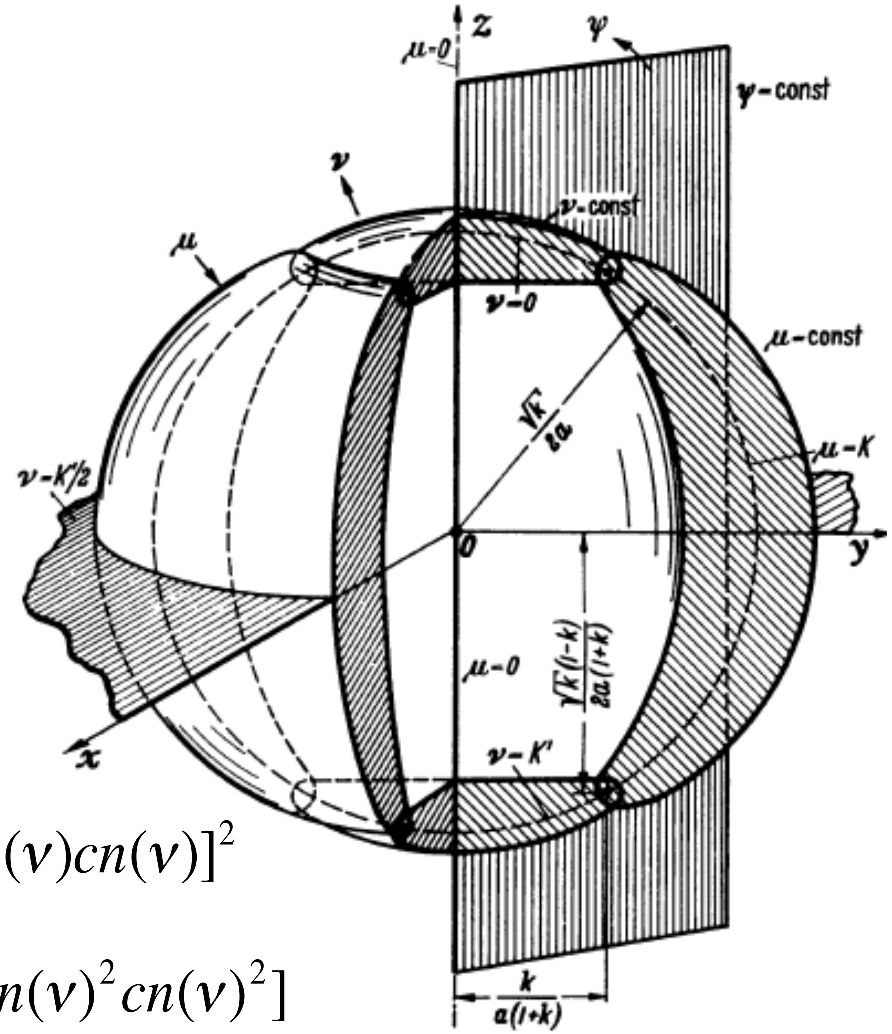
$0 \leq \mu \leq \mathbf{K}$ $0 \leq \nu \leq \mathbf{K'}$ $0 \leq \phi \leq 2\pi$

And **K** and **K'** are the complete elliptic integral with respectively modules k and k'



Let me to introduce the Cap-Cyclides Coordinates System

$$\left\{\begin{aligned} x &= \frac{\Lambda}{a_s\Gamma} dn(\nu,k')\, sn(\mu,k)\cos\varphi \\ y &= \frac{\Lambda}{a_s\Gamma} dn(\nu,k')\, sn(\mu,k)\sin s\varphi \\ z &= \frac{k^{0.5}\Pi}{2a_s\Gamma} \end{aligned}\right\}$$



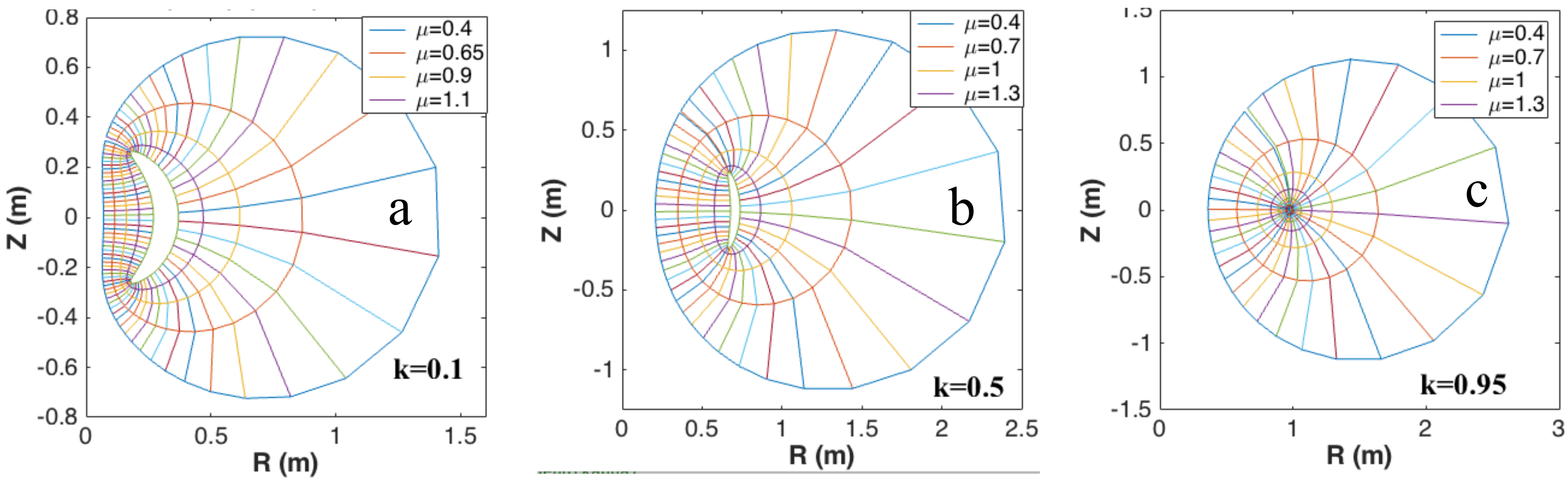
Where

$$\Lambda = 1 - dn^2(\mu)sn^2(\nu)$$

$$\Gamma = sn^2(\mu)dn^2(\nu) + [(\Lambda/k^{0.5}) + cn(\mu)dn(\mu)sn(\nu)cn(\nu)]^2$$

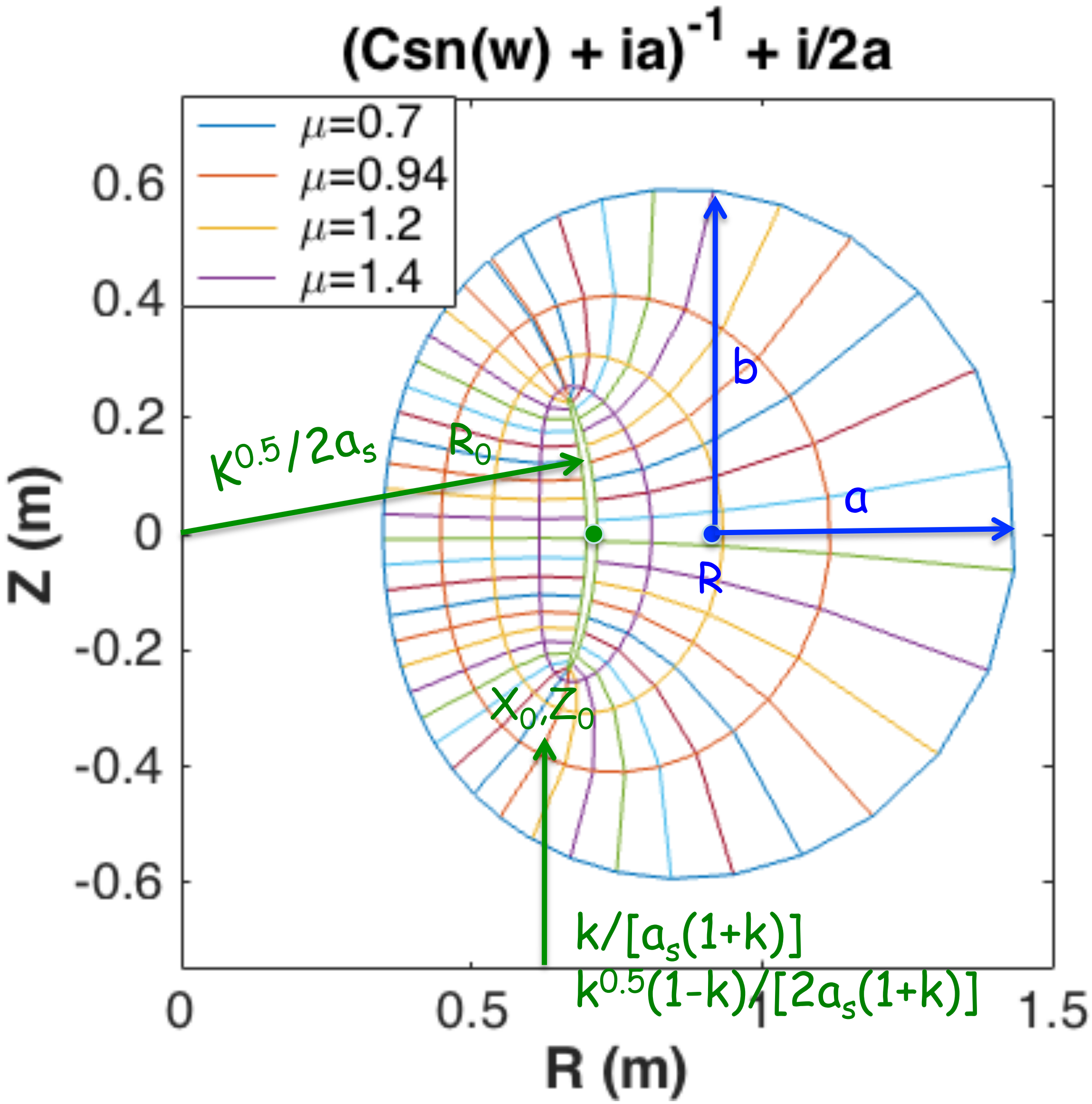
$$\Pi = (\Lambda^2/k) - [sn^2(\mu)dn^2(\nu) + cn(\mu)^2 dn(\mu)^2 sn(\nu)^2 cn(\nu)^2]$$

Laplace Equation Cap-Cyclide Coordinates



For the elliptical integral module k → 1 and/or for the "radial" coordinate μ → 0 (R→∞) the coordinates goes to the circular toroidal one

Laplace Equation Cap-Cyclide Coordinates



Center of Cap-Cyclide coordinate system is

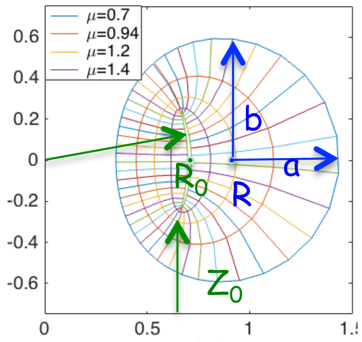
$R_0=k^{0.5}/2a_s$

$Z_0=\pm k^{0.5}(1-k)/[2a_s(1+k)]$

$X_0=k^{0.5}/[a_s(1+k)]$

that tends to the center of the circular toroidal system for $k \to 1$
[$R_0=1/2a_s$; $Z_0=0$]

We can define the local aspect ratio a/R and the local elongation b/a



After some algebra, we can get local aspect ratio

$$R = \frac{k^{0.5}\left(k\ sn^2\mu + 1\right)}{2a_s(1+k)\ sn\ \mu} \qquad a = \frac{k^{0.5} cn\ \mu\ dn\ \mu}{2a_s(1+k)\ sn\ \mu}$$

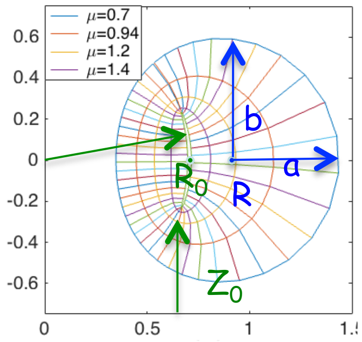
For the large aspect ratio case μ→ **K** we have $R = R_0 = \frac{k^{0.5}}{2a_s}\ ;\ a = 0$

Instead for μ→ 0 for κ → 1 we get

$$\frac{R}{a} = \frac{\left(k\ sn^2\mu + 1\right)}{cn\ \mu\ dn\ \mu} \approx \cosh\ (\mu)$$

Note that this behaviour is similar to the toroidal limit where $R_{Tor}/a_{Tor} = \cosh\mu$

Laplace Equation Cap-Cyclide Coordinates



The evaluation of the local elongation is a bit more complicated. We proceed by the fact that the maximum and minimum zeta value happens when, respectively, the two internal/external up and down radial coordinates coincides

Eventually, after some heavy algebra, we get

$$z_{\max} = b = \frac{k^{0.5}\left(1-k\ sn^2\mu\right)}{2a_s\left(1+k\right)sn\ \mu} \Rightarrow \frac{b}{a} = \frac{\left(1-k\ sn^2\mu\right)}{cn\ \mu\ dn\ \mu}$$

$$\Rightarrow\ \mu = 0\ \ (k=0;k=1)\ \Rightarrow \frac{b}{a} = 1$$ or $$\Rightarrow\ \mu = K\ \ (k=0;k\neq 1)\ \Rightarrow \frac{b}{a} = \infty$$

$$\Rightarrow\ \mu = K\ \ (k=1)\ \Rightarrow \frac{b}{a} = 1$$

Cap-Cyclide Laplace Equation

Laplace equation for an Euclidean space and for an orthogonal system is

$$\frac{1}{\sqrt{g}}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\sqrt{g}}{g_{ii}}\frac{\partial\phi}{\partial x_i}\right)=0$$

That assuming $\phi = U_1(x_1)\ldots U_n(x_n)$ becomes

$$\sum_{i=1}^{n}\frac{1}{U_i}\frac{\partial}{\partial x_i}\left(\frac{\sqrt{g}}{g_{ii}}\frac{\partial U_i}{\partial x_i}\right)=0$$

That using the separability conditions can be rewritten as[1]

$$\sum_{i=1}^{n}\frac{1}{g_{ii}}\left[\frac{1}{f_iU_i}\frac{d}{dx_i}\left(f_i\frac{dU_i}{dx_i}\right)+\sum_{j=1}^{n}\alpha_j\Phi_{ij}(x_i)\right]=0$$

That is satisfied only when any single addend is equal 0

$$\frac{1}{f_iU_i}\frac{d}{dx_i}\left(f_i\frac{dU_i}{dx_i}\right)+\sum_{j=1}^{n}\alpha_j\Phi_{ij}(x_i)=0$$

[1]P. Moon and D. E. Spencer, Jour. Franklin Inst., 253, (1952) 585

Cap-Cyclide Laplace Equation

Coming back to the Laplace Equation we can write the solution as

$$\phi(\mu,\nu,\varphi) = \left(\frac{\Gamma}{\Lambda}\right)^{0.5} M(\mu)N(\nu)F(\varphi)$$

That give the following set of independent equation

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{cn\ \mu\ dn\ \mu}{sn\ \mu}\dfrac{d\ M}{d\mu} + \left[k^2 sn^2\mu - \alpha_2 - \alpha_3\left(k^2 sn^2\mu + \dfrac{1}{sn^2\mu}\right)\right]M = 0 \\ \dfrac{d^2N}{d\nu^2} - \dfrac{k'^2 sn\ \nu\ cn\ \nu}{dn\ \nu}\dfrac{d\ N}{d\nu} + \left[-dn^2\nu + \alpha_2 + \alpha_3\left(dn^2\nu + \dfrac{k^2}{dn^2\nu}\right)\right]N = 0 \\ \dfrac{d^2F}{d\varphi^2} + \alpha_3 F = 0 \end{cases}$$

Cap-Cyclide Laplace Equation

If in the equation for $M(\mu)$ we substitute $sn^2\mu = z_1$ we get

$$\frac{d^2M}{dz_1^2} + \frac{dM}{dz_1}\frac{\left\{\left(1-k^2z_1\right)\left[2\left(1-z_1\right)-z_1\right]-k^2z_1\left(1-z_1\right)\right\}}{2z_1\left(1-k^2z_1\right)\left(1-z_1\right)} + \frac{\left[k^2z_1-\alpha_2-\alpha_3\left(k^2z_1+\frac{1}{z_1}\right)\right]\frac{M}{4}}{z_1\left(1-k^2z_1\right)\left(1-z_1\right)} = 0$$

That can be written as

$$\frac{d^2M}{dz_1^2} + \frac{1}{2}\left[\frac{1}{z_1-a_1}+\frac{1}{z_1-a_2}+\frac{2}{z_1-a_3}\right]\frac{dM}{dz_1} + \frac{1}{4}\left[\frac{A_0+A_1z_1+A_2z_1^2}{\left(z_1-a_1\right)\left(z_1-a_2\right)\left(z_1-a_3\right)^2}\right]M = 0$$

With $a_1=1$; $a_2=1/k^2$; $a_3=0$; $A_0=-\alpha_3/k^2$; $A_1=-\alpha_2/k^2$; $A_2=1-\alpha_3$

Cap-Cyclide Laplace Equation

If in the equation for for the polidal angular part N(υ) we substitute $dn^2\upsilon = z_2$ we get

$$\frac{d^2N}{dz_2^2}+\frac{dN}{dz_2}\frac{\left\{\left(z_2-k^2\right)\left[2\left(1-z_2\right)-z_2\right]+z_2\left(1-z_2\right)\right\}}{2z_2\left(z_2-k^2\right)\left(1-z_2\right)}+\frac{\left[-z_2+\alpha_2+\alpha_3\left(z_2+\frac{k^2}{z_2}\right)\right]\frac{N}{4}}{z_2\left(z_2-k^2\right)\left(1-z_2\right)}=0$$

That can be written as

$$\frac{d^2N}{dz_2^2}+\frac{1}{2}\left[\frac{-1}{1-z_2}+\frac{1}{z_2-k^2}+\frac{2}{z_2}\right]\frac{dN}{dz_2}+\frac{1}{4}\left[\frac{-z_2^2+\alpha_2 z_2+\alpha_3 z_2^2+\alpha_3 k^2}{z_2^2\left(1-z_2\right)\left(z_2-k^2\right)}\right]N=0$$

Cap-Cyclide Laplace Equation

That is a particular case of the general Bôcher[1] equation

$$\frac{d^2\tilde{Z}}{dz^2} + \frac{d\tilde{Z}}{dz}P(z) + Q(z)\tilde{Z} = 0$$

Where

$$P(z) = \frac{1}{2}\left[\frac{m_1}{z-a_1} + \frac{m_2}{z-a_2} + \ldots + \frac{m_{n-1}}{z-a_{n-1}}\right]$$

$$Q(z) = \frac{1}{4}\left[\frac{A_0 + A_1 z + \ldots + A_l z^l}{(z_1-a_1)^{m_1}(z_1-a_2)^{m_2}\ldots(z_1-a_3)^{m_{n-1}}}\right]$$

With $a_1=1$; $a_2=1/k^2$; $a_3=0$; $A_0=-\alpha_3/k^2$; $A_1=-\alpha_2/k^2$; $A_2=1-\alpha_3$

[1]M. Bôcher, "Über die Reihenentwickelungen der Potential theorie", Göttingen, B. G.Teubner., 1891

Cap-Cyclide Laplace Equation

These equations are known as Wangerin[1] equations

$$\frac{d^2\tilde{Z}}{dz^2}+\frac{1}{2}\left[\frac{1}{z-1}+\frac{1}{z-c}+\frac{2}{z}\right]\frac{d\tilde{Z}}{dz}+\frac{1}{4}\left[\frac{-q^2c-p^2cz+\left(1-q^2\right)z^2}{\left(z-1\right)\left(z_1-c\right)z^2}\right]\tilde{Z}=0$$

This equation has three singularities with poles in z=1,c,0 of the order {1,1,2). Another pole is for z→∞ of order 2. Consequently the Wangerin equation is characterized by poles {1,1,2,2}.
The equation admits a solution that depends from z, c, p, q and the general solution can be written as

$$\phi(\mu,\nu,\varphi)=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\left[A_{p'}^{0}\mathcal{D}_{p'}^{q}(k,sn^2\mu)+B_{p'}^{0}\mathcal{F}_{p'}^{q}(k,sn^2\mu)\right]\left[A_{p'}^{1}\mathcal{D}_{p'}^{q}(1/k,dn^2\nu)+B_{p'}^{1}\mathcal{F}_{p'}^{q}(1/k,dn^2\nu)\right]$$

[1]A. Wangerin, "Theorie des Potentials und der Kugelfunktionen", Leipzig, G. J. Göschen'sche, 1909

Cap-Cyclide Grad-Shafranov Equation

We can use a complete different approach that will bring to an equation of the Wangerin type.

$$\left(\Delta A\right)_{\varphi} = -\left(\nabla^{*2}\Psi\right)/R$$

Where ∇^{*2} is the Grad-Shafranov operator

In axis symmetry the symmetrical component of the Laplacian operator is

$$\frac{\partial^2 A_\varphi}{\partial u_1^2} + \frac{\partial^2 A_\varphi}{\partial u_2^2} + \frac{1}{2g_{33}}\left(\frac{\partial g_{33}}{\partial u_1}\frac{\partial A_\varphi}{\partial u_1} + \frac{\partial g_{33}}{\partial u_2}\frac{\partial A_\varphi}{\partial u_2}\right) +$$

$$+\frac{A_\varphi}{2g_{33}}\left\{\left(\frac{\partial^2 g_{33}}{\partial u_1^2} + \frac{\partial^2 g_{33}}{\partial u_2^2}\right) - \frac{1}{g_{33}}\left[\left(\frac{\partial g_{33}}{\partial u_1}\right)^2 + \left(\frac{\partial g_{33}}{\partial u_2}\right)^2\right]\right\} = 0$$

Cap-Cyclide Grad-Shafranov Equation

In our Cap-Cyclide coordinates this become

$$\frac{\partial}{\partial\mu}\left[\frac{\partial A_\varphi}{\partial\mu}+\left(\frac{1}{f_1}\frac{\partial f_1}{\partial\mu}-R^2\frac{\partial R^{-2}}{\partial\mu}\right)A_\varphi\right]+\frac{\partial}{\partial\nu}\left[\frac{\partial A_\varphi}{\partial\nu}+\left(\frac{1}{f_2}\frac{\partial f_2}{\partial\nu}-R^2\frac{\partial R^{-2}}{\partial\nu}\right)A_\varphi\right]=0$$

Where here $R=(\Lambda/a_s\Gamma)^{1/2}$ is the quasi separation factor of the Laplace equation and $f_1(\mu)=\mathrm{sn}\mu$, $f_2(\upsilon)=\mathrm{dn}\upsilon$

This equation is equal to the Laplace equation PLUS a non differential term in A_φ

Cap-Cyclide Grad-Shafranov Equation

In our case of the Caps-Cyclide coordinates this becomes

$$\left[\frac{\partial}{\partial\mu}\left(\frac{1}{sn\mu}\frac{\partial sn\mu}{\partial\mu}-\frac{\Lambda}{\Gamma}\frac{\partial}{\partial\mu}\frac{\Gamma}{\Lambda}\right)+\frac{\partial}{\partial\nu}\left(\frac{1}{dn\nu}\frac{\partial dn\nu}{\partial\nu}-\frac{\Lambda}{\Gamma}\frac{\partial}{\partial\nu}\frac{\Gamma}{\Lambda}\right)\right]=$$

$$=-\left(k^2 sn^2\mu+\frac{1}{sn^2\mu}\right)+\left(dn^2\nu+\frac{k^2}{dn^2\nu}\right)$$

We know that the solution for the Vector Laplace equation in axis symmetry is separable and can be written as

$$A_\varphi(\mu,\nu)=\frac{M(\mu)N(\nu)}{R(\mu,\nu)}=\left(\frac{\Gamma}{\Lambda}\right)^{1/2}M(\mu)N(\nu)$$

Cap-Cyclide Grad-Shafranov Equation

Consequently it can be written a system of independent equation similar to the Laplace ones with $q^2=\alpha_3=0$ (axis symmetry), but with an additional terms in $A\varphi$

$$\left[\begin{array}{l} \dfrac{d^2M}{d\mu^2} + \dfrac{cn\mu \quad dn\mu}{sn\mu}\dfrac{dM}{d\mu} + \left(\cancel{k}^2\cancel{sn}^2\mu - \alpha_2 - \cancel{k}^2\cancel{sn}^2\mu + \dfrac{1}{sn^2\mu}\right)M = 0 \\ \dfrac{d^2N}{d\nu^2} - k^2\dfrac{sn\nu \quad cn\nu}{dn\nu}\dfrac{dN}{d\nu} + \left(-\cancel{dn}^2\nu + \alpha_2 + \cancel{dn}^2\nu + \dfrac{k^2}{sn^2\nu}\right)N = 0 \end{array}\right.$$

But this is exactly the set of equations for the 3D scalar Laplace equation with $q^2=\alpha_3=1$
So the solution of the axial component of the vectorial Laplace equation for the vector potential A is the same solution that for the scalar Laplace equation with q=1 and $\varphi=0$

Cap-Cyclide Grad-Shafranov Equation

If now we remember that the flux function is defined as

$$\Psi = \oint A_\varphi d\ell$$

We have that

$$\Psi = 2\pi \quad g_{33}^{1/2} A_\varphi(\mu,\nu) = 2\pi \quad g_{33}^{1/2} \frac{M(\mu)N(\nu)}{R(\mu,\nu)} = 2\pi \quad g_{33}^{1/2}\left(\frac{\Gamma}{\Lambda}\right)^{1/2} M(\mu)N(\nu)$$

And eventually

$$\Psi(\mu,\nu) = \frac{2\pi}{a_s}\left(\frac{\Lambda}{\Gamma}\right)^{1/2} sn\mu \quad dn\nu\left[A_{p'}^{0}\mathcal{D}_{p'}^{1}(k,sn^2\mu) + B_{p'}^{0}\mathcal{F}_{p'}^{1}(k,sn^2\mu)\right]\left[A_{p'}^{1}\mathcal{D}_{p'}^{1}(1/k,dn^2\nu) + B_{p'}^{1}\mathcal{F}_{p'}^{1}(1/k,dn^2\nu)\right]$$

Conclusion

Mixing up some different music styles, i.e. playing with the math the analytical solution of Grad-Shafranov Equation has been written in a toroidal elliptical prolate geometry. This will allow to develop a reconstructive equilibrium code based on the natural elongated plasma geometry