An adjoint method for gradient-based optimization of stellarator coil shapes

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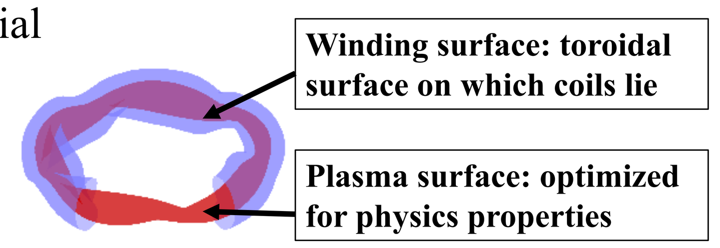
Introduction

- For the feasibility of the stellarator, we must design simple coils which can be produced within engineering tolerances
- Typically coil design process assumes a known target plasma surface (chosen for MHD stability, neoclassical confinement, rotational transform profiles, etc.)
- Characteristics of "good" coils
  - Minimize error in reproducing desired magnetic surfaces
  - Coil-plasma distance to allow for a blanket and plasma-facing components, and divertor
  - Sufficient coil-coil distance for diagnostic and maintenance
  - Low curvature to allow for finite thickness of conducting material



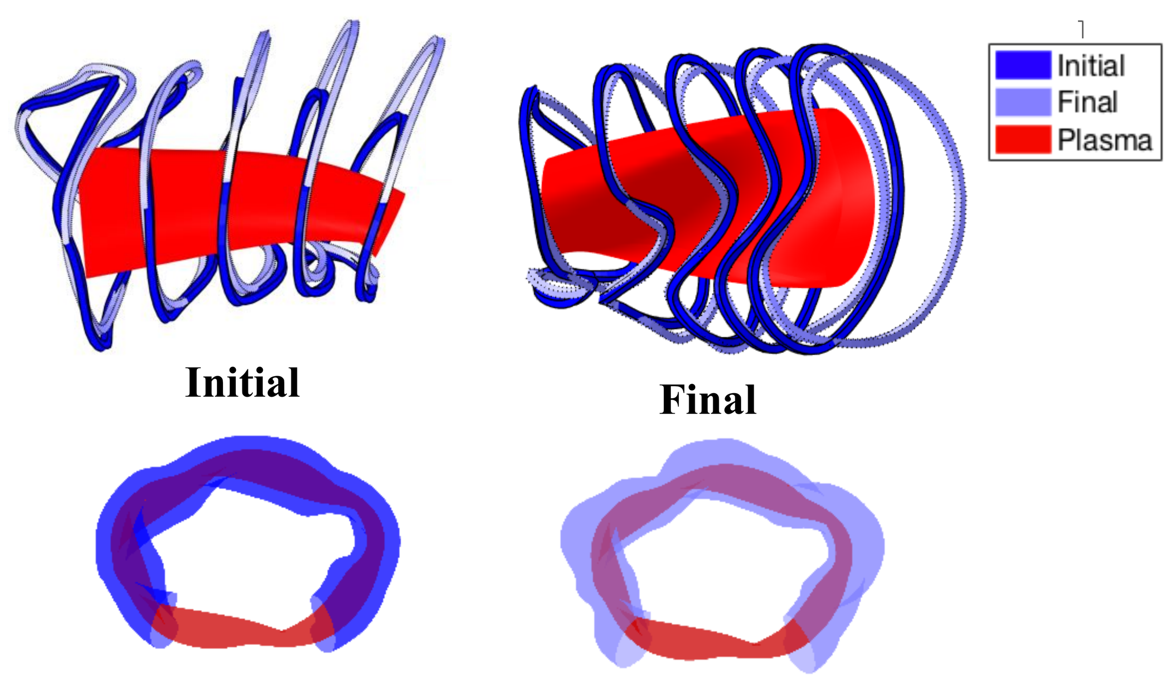
- Our approach: optimize winding surface using a current potential approach (used in W7-X and NCSX initial coil studies [1,2])
  - Convex problem solved to obtain coils for on a winding surface
  - Optimize winding surface to allow for "good" coils
  - Adjoint methods are used to compute derivatives for gradient-based optimization - reduces function evaluations required

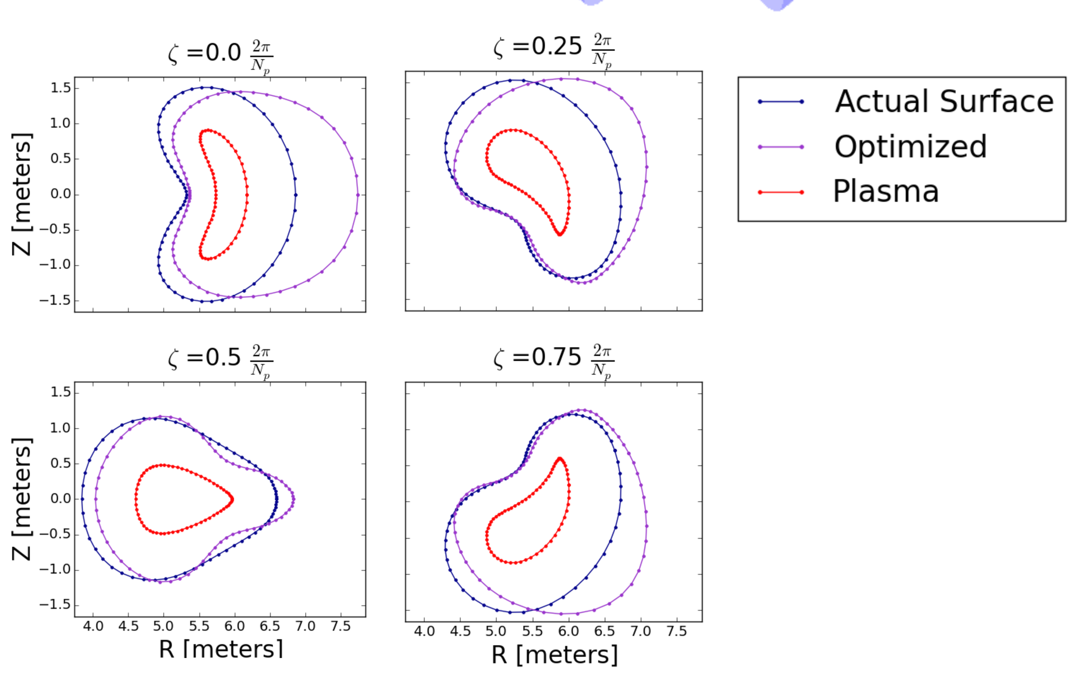
W7-X Optimization Results

We demonstrate this optimization method, beginning with the actual W7-X and HSX winding surfaces

- Simultaneously reduce the normal field error and increase coil-plasma distance in convex regions
- Coils computed on optimal surface are less complex

	Initial	Optimized
Min. coil-coil distance [m]	0.223	0.271
Max curvature [m^{-1}]	9.01	4.84
Max toroidal extent [rad.]	0.222	0.197
Normal field error [T^{2}m^2]	0.115	0.0711



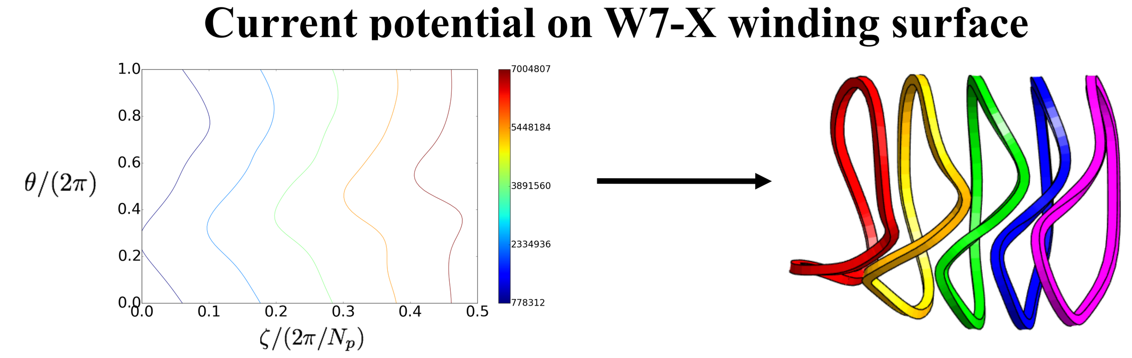


REGCOIL [3] Overview

Given fixed coil and plasma surfaces, REGCOIL is used to quickly obtain coil shapes

- Divergentless surface current described by current potential, Φ
$$\boldsymbol{K} = \boldsymbol{n} \times \nabla \Phi$$
- Contours of Φ (streamlines of K) give discrete coil shapes

Current potential on W7-X winding surface



Optimization of current potential

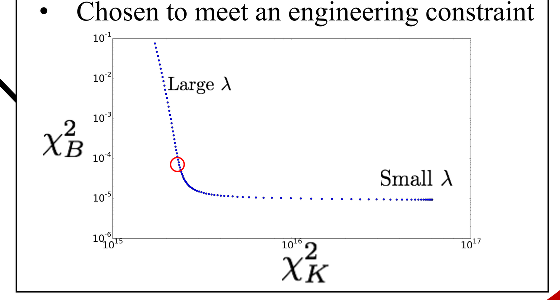
- Secular terms of Φ fixed by required flux
$$\Phi = \Phi_{\text{sv}} + \frac{G\zeta}{2\pi} + \frac{I\theta}{2\pi}$$
- Single-valued current potential is Fourier decomposed
$$\Phi_{\text{sv}} = \sum_{m,n} \Phi_{mn} \sin(m\theta - n\zeta)$$
- Objective function minimized using linear least-squares

Increased coil-coil spacing
$$\chi_K^2 = \int_{\text{coil}} d^2 A\, K^2$$

Our task: $\min_{\Phi} \chi^2 = \chi_B^2 + \lambda \chi_K^2$

Fidelity in reproducing plasma surface
$$\chi_B^2 = \int_{\text{plasma}} d^2 A\, B_n^2$$

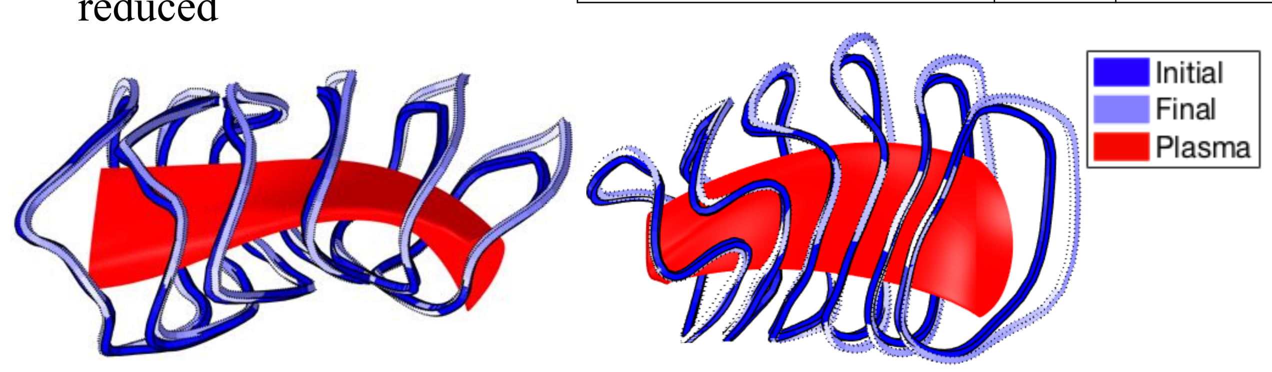
Regularization parameter
- Chosen to meet an engineering constraint

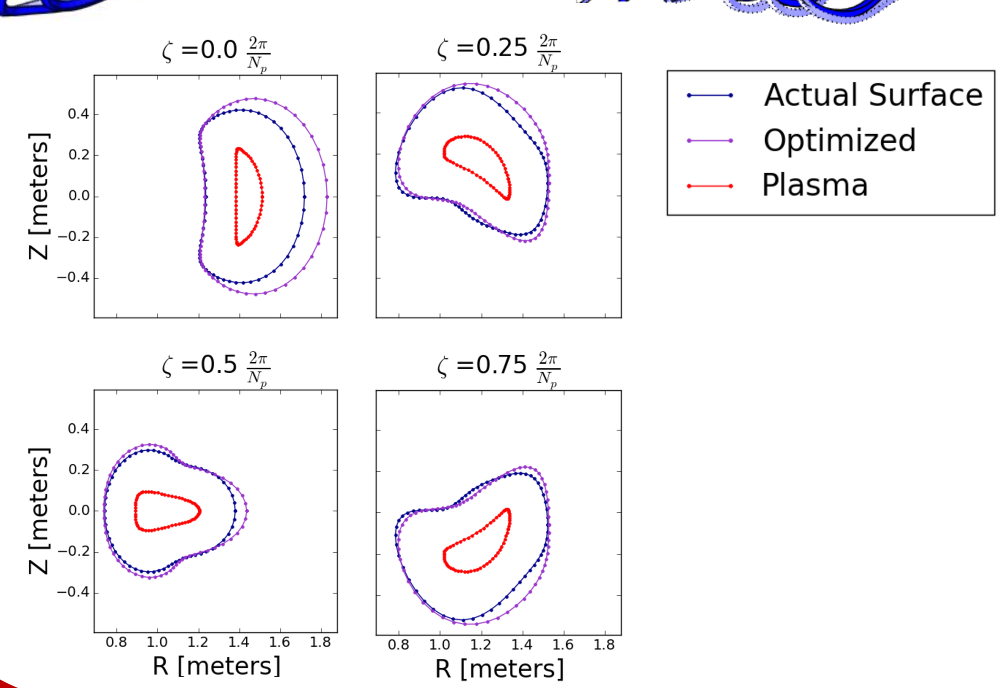


HSX Optimization Results

- Optimization begins with HSX winding surface
- Normal field error and coil complexity reduced

	Initial	Optimized
Min. coil-coil distance [m]	0.0850	0.0853
Max curvature [m^{-1}]	33.4	25.8
Max toroidal extent [rad.]	0.530	0.505
Normal field error [T^{2}m^2]	1.53e-5	1.47e-5





Winding Surface Optimization

We extend the REGCOIL method, allowing variation of the winding surface, to optimize coils in 3D space

- Optimization space: Fourier coefficients of winding surface, $\Omega = \{r_{mn}^c, z_{mn}^s\}$

$$x = \sum_{m,n} r_{mn}^c \cos(m\theta - n\zeta)\cos\zeta$$
$$y = \sum_{m,n} r_{mn}^c \cos(m\theta - n\zeta)\sin\zeta$$
$$z = \sum_{m,n} z_{mn}^s \sin(m\theta - n\zeta)$$

Cartesian components of stellarator-symmetric winding surface

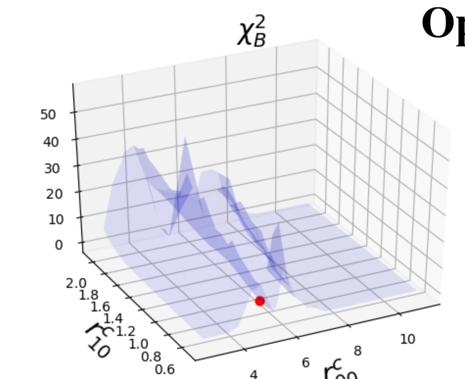
Objective Function

$$f = \chi_B^2 + \alpha_K \|\boldsymbol{K}\|_2 - \alpha_V V_{\text{coil}}^{1/3} + \alpha_S S_p$$

- V_{coil} = proxy for coil-plasma distance
- S_p = measure of Fourier spectrum condensation [4]
$$S_p = \sum_{m,n} m^p \left((r_{mn}^c)^2 + (z_{mn}^s)^2 \right)$$
- χ_B^2 = proxy for physics properties
- $\|\boldsymbol{K}\|_2$ = proxy for coil complexity

Constraints

- $d_{\text{coil-plasma}} \geq d_{\text{min}}^{\text{target}}$ - ensures sufficient coil-plasma distance
- K_{max} is fixed to choose regularization, λ (roughly equivalent to fixed coil-coil spacing)



Optimization Algorithm

- REGCOIL used to obtain surface current density on each winding surface
- Constrained, gradient-based non-linear optimization using a conservative convex separable approximation algorithm [5]
- Gradients of objective function obtained using linear adjoint method

Shape Gradients for Sensitivity Analysis [6]

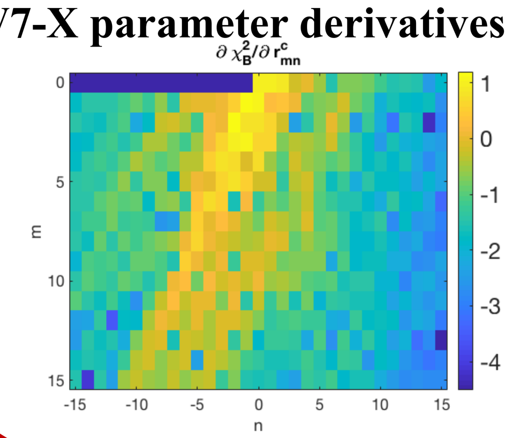
We use the shape gradient to compute local sensitivity to winding surface perturbations

- Consider the perturbation of a shape, Γ, by a velocity field $\delta \boldsymbol{r}$
$$\Gamma_\epsilon = \{\boldsymbol{r}_0 + \epsilon \delta \boldsymbol{r}(\boldsymbol{r}_0) : \boldsymbol{r}_0 \in \Gamma\}$$
- Definition of the shape derivative of the functional of the shape
$$\delta f(\Gamma, \delta \boldsymbol{r}) = \lim_{\epsilon \to 0} \frac{f(\Gamma_\epsilon) - f(\Gamma)}{\epsilon}$$
- For many functionals, the shape derivative can be written in terms of normal displacements of a surface
$$\boxed{\delta f(\Gamma, \delta \boldsymbol{r}) = \int_{\partial \Gamma} d^2 A\, S \delta \boldsymbol{r} \cdot \boldsymbol{n}}$$

Shape gradient, S, provides local sensitivity

- Shape gradient is Fourier decomposed
$$S = \sum_j S_j \cos(m_j \theta - n_j \zeta)$$

W7-X parameter derivatives



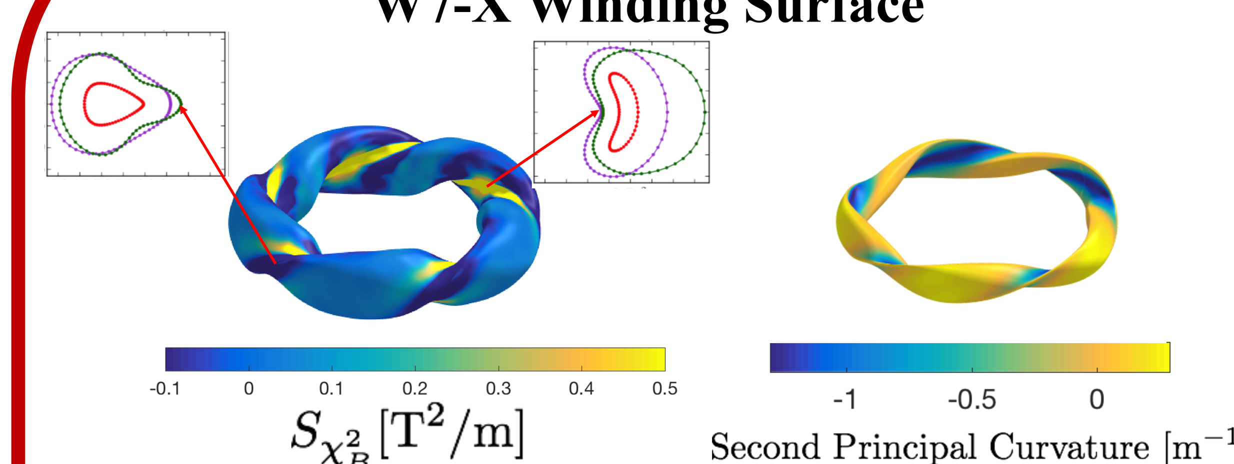
Using known parameter derivatives, this takes the form of a (generally not square) linear system, solved using a Moore-Penrose pseudoinverse
$$\frac{\partial f}{\partial \Omega_j} = \sum_j D_{ij} S_j$$

Adjoint Method for a Linear System

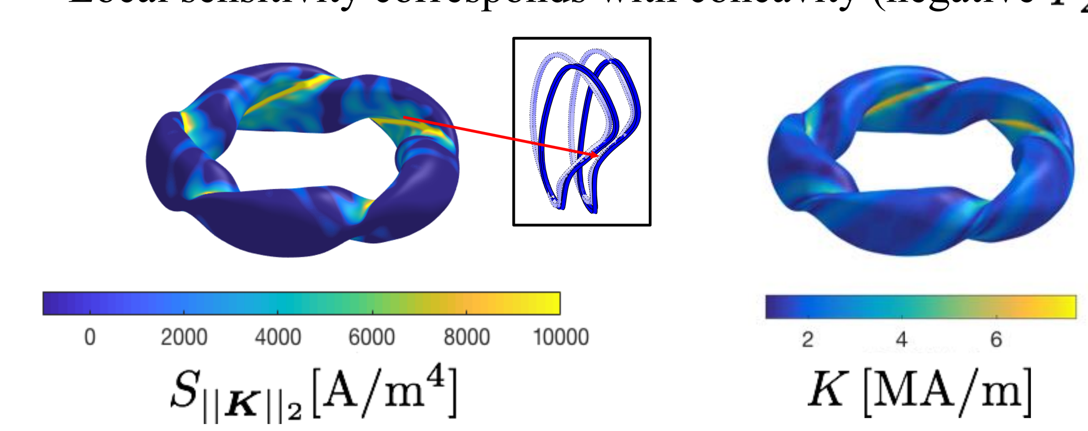
The adjoint method allows us to compute gradients with respect to many input parameters efficiently

- State variable, x, satisfies linear forward equation and depends on set of input parameters, $\Omega = \{\Omega_i\}$
$$\boldsymbol{A}(\Omega_i)\boldsymbol{x} = \boldsymbol{b}(\Omega_i)$$
- Need to compute gradient of objective function, $f(\Omega, x(\Omega))$
$$\left.\frac{\partial f(\Omega, x(\Omega))}{\partial \Omega}\right|_{Ax=b} = \left.\frac{\partial f}{\partial \Omega}\right|_x + \left(\frac{\partial f}{\partial x}\right) \cdot \left.\frac{\partial x}{\partial \Omega}\right|_{Ax=b}$$
- Consider perturbations of the linear system
$$\frac{\partial \boldsymbol{A}}{\partial \Omega_i}\boldsymbol{x} + \boldsymbol{A}\frac{\partial \boldsymbol{x}}{\partial \Omega_i} = \frac{\partial \boldsymbol{b}}{\partial \Omega_i} \rightarrow \left.\frac{\partial \boldsymbol{x}}{\partial \Omega_i}\right|_{\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}} = \boldsymbol{A}^{-1}\left(\frac{\partial \boldsymbol{b}}{\partial \Omega_i} - \frac{\partial \boldsymbol{A}}{\partial \Omega_i}\boldsymbol{x}\right)$$
$$\left.\frac{\partial f(\Omega, \boldsymbol{x}(\Omega))}{\partial \Omega_i}\right|_{\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}} = \left.\frac{\partial f}{\partial \Omega_i}\right|_{\boldsymbol{x}} + \left(\frac{\partial f}{\partial \boldsymbol{x}}\right) \cdot \left(\boldsymbol{A}^{-1}\left(\frac{\partial \boldsymbol{b}}{\partial \Omega_i} - \frac{\partial \boldsymbol{A}}{\partial \Omega_i}\boldsymbol{x}\right)\right)$$
- Use adjoint property for inner product
$$\left.\frac{\partial f(\Omega, \boldsymbol{x}(\Omega))}{\partial \Omega_i}\right|_{\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}} = \left.\frac{\partial f}{\partial \Omega_i}\right|_{\boldsymbol{x}} + \underbrace{\left((\boldsymbol{A}^T)^{-1}\frac{\partial f}{\partial \boldsymbol{x}}\right)}_{\boldsymbol{q}} \cdot \left(\frac{\partial \boldsymbol{b}}{\partial \Omega_i} - \frac{\partial \boldsymbol{A}}{\partial \Omega_i}\boldsymbol{x}\right)$$
- Compute solution to adjoint equation
$$\boldsymbol{A}^T \boldsymbol{q} = \frac{\partial f}{\partial \boldsymbol{x}}$$
- With 2 linear solves (forward and adjoint), can compute gradient with respect to many input parameters
- Assuming linear solve is the computational bottleneck, scales independent of N_Ω and linear in number of objective functions

Shape Gradients of Functionals of the W7-X Winding Surface



- Identifies where optimization increased/decreased coil-plasma
- Local sensitivity corresponds with concavity (negative P_2)



- Identifies regions requiring small coil-coil spacing

References

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[6] M. Landreman and E. Paul, (2018), arXiv:1803.03187 [plasm-ph].