

A gyrokinetic model for the tokamak periphery

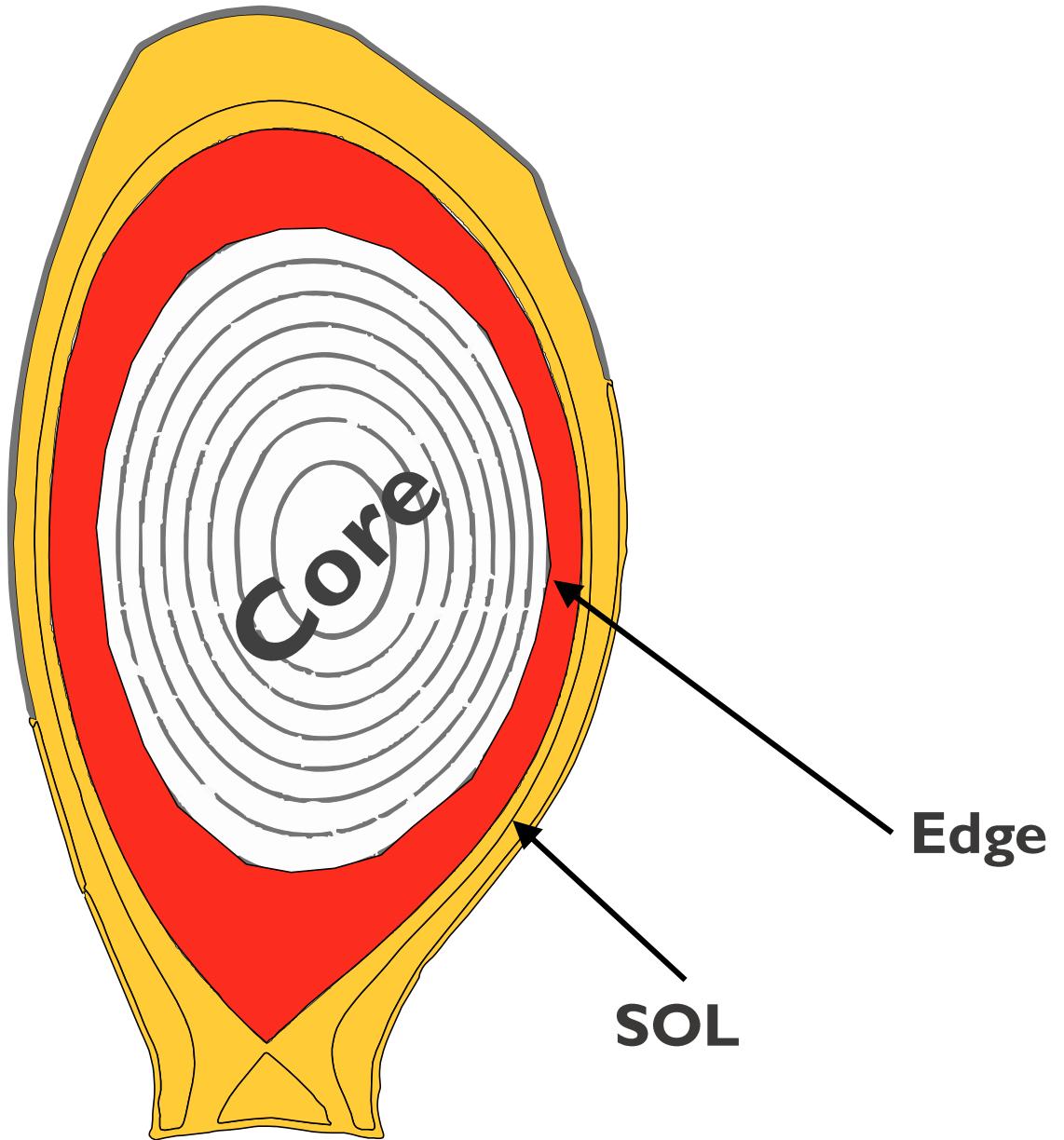
R. Jorge^{1,2}, B. Frei¹, A. Baillod¹, P. Ricci¹, N. F. Loureiro³



SWISS PLASMA
CENTER



The Tokamak Periphery = Edge + SOL



- Core boundary conditions
- Heat exhaust
- Plasma fueling and ashes removal
- Impurity control

Properties of Periphery Turbulence

- Low-frequency $\omega \ll \Omega_i$

- Large Scale Fluctuations

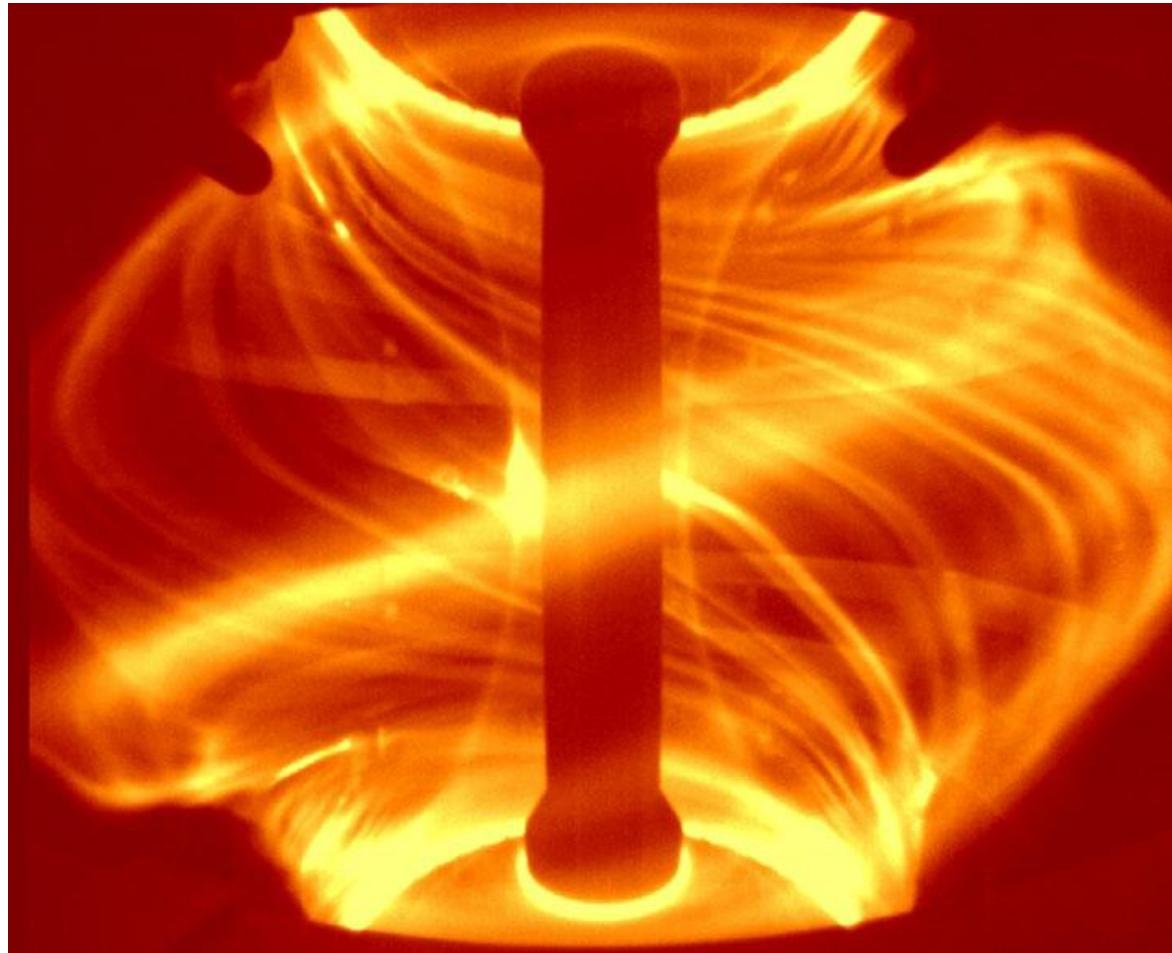
$$k_{\perp} \rho_i \ll 1 \quad \frac{e\phi}{T_e} \sim 1$$

- Small Scale Fluctuations

$$k_{\perp} \rho_i \sim 1 \quad \frac{e\phi}{T_e} \ll 1$$

- Wide range of temperatures
and densities

$$T_e \sim 1 - 10^3 \text{ eV} \quad n \sim 10^{18} - 10^{20} \text{ m}^{-3}$$



Properties of Periphery Turbulence

- Low-frequency $\omega \ll \Omega_i$

- Large Scale Fluctuations

$$k_{\perp} \rho_i \ll 1 \quad \frac{e\phi}{T_e} \sim 1$$



Gyrokinetic Theory

$$\epsilon = k_{\perp} \rho_i \frac{e\phi}{T_e} \ll 1$$

- Small Scale Fluctuations

$$k_{\perp} \rho_i \sim 1 \quad \frac{e\phi}{T_e} \ll 1$$

- Wide range of temperatures

and densities

$$T_e \sim 1 - 10^3 \text{ eV} \quad n \sim 10^{18} - 10^{20} \text{ m}^{-3}$$

→ Arbitrary Collisionality

First Heroic Steps On The Way To An Edge Model

	Qin et al. 2006¹	Hahm et al. 2009²	Dimitis et al. 2012³
Small Scale Fluctuations	EM $O(\epsilon^2)$	EM $O(\epsilon^2)$	EM $O(\epsilon^2)$
Large Scale Fluctuations	EM $O(\epsilon)$	ES $O(\epsilon^2)$	ES $O(\epsilon^2)$
Poisson's Eq.	$\int(\dots) d^3v$	Long-Wavelength Limit	$\int(\dots) d^3v$
Collisions	No	No	No

¹Qin et al., Contri. Plasma Phys. 46 (2006)

²Hahm et al., Physics of Plasmas 16 (2009)

³Dimitis, Physics of Plasmas 19 (2012)

Our Model

Retain

- Both large scale (full-F) and small scale fluctuations
- Second order
- Full Coulomb collisions
- Numerical efficiency

How we proceed

Single Particle
Dynamics



Gyrokinetic
Theory



Moment
Hierarchy

Single Particle Dynamics - Ordering Assumptions

Magnetized Plasma

$$\frac{\omega}{\Omega_i} \sim \epsilon k_{\perp} \rho_i$$

$$\frac{\rho_i}{L_p} \sim \epsilon$$

Fluctuation Scales and Amplitude

$$k_{\perp} \rho_i \frac{e\phi}{T_e} = \epsilon \ll 1$$

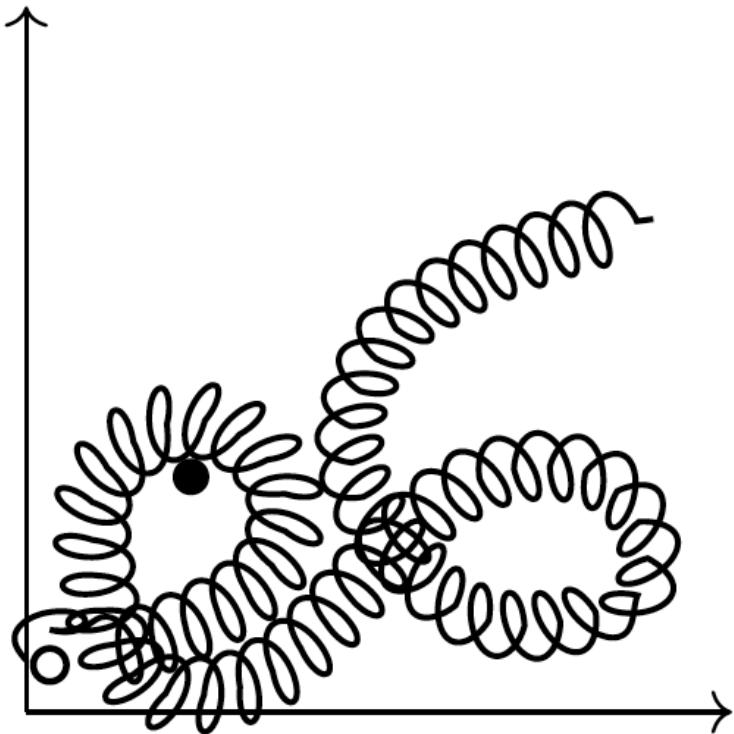
Electromagnetic

$$v_{\parallel} A_{\parallel} \sim \phi$$

Arbitrary Collisionalities
(still magnetized)

$$\frac{\nu_{ei}}{\Omega_i} \lesssim \epsilon$$

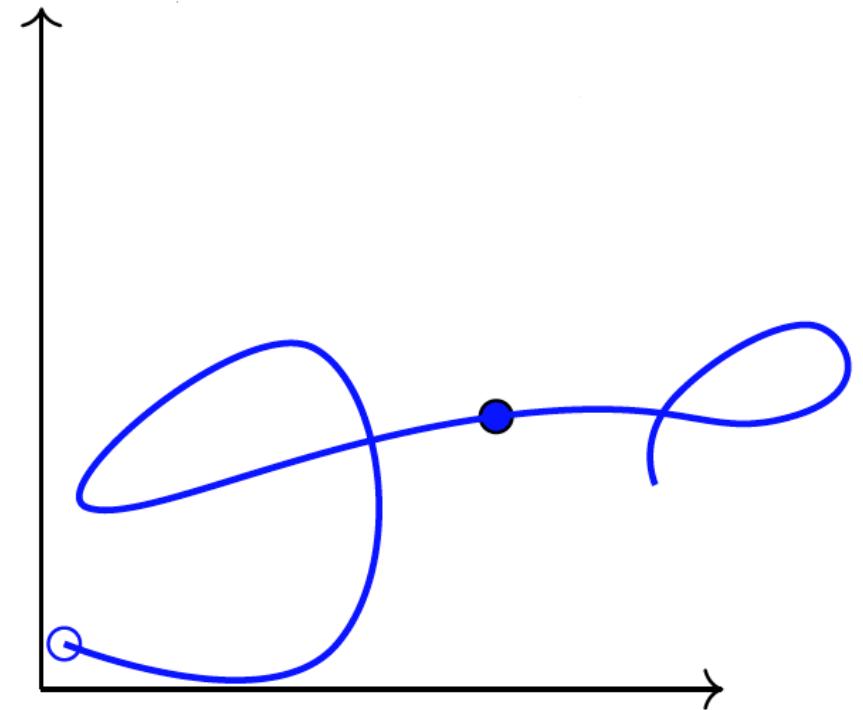
Single Particle Dynamics – Hamiltonian Perturbation Theory



Particle Lagrangian

$$L(\mathbf{x}, \mathbf{v})$$

2-Step Phase-Space
Reduction



Lagrangian

$$\Gamma(\mathbf{R}, v_{\parallel}, \mu)$$

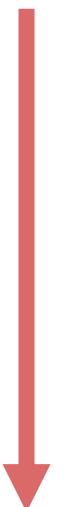
(cyclic coordinate θ , conserved μ)

Single Particle Dynamics – First Step

Large Scales $\epsilon \sim k_{\perp} \rho_i \ll 1$, $\frac{e\phi}{T_e} \sim 1$

Start from

$$L = q\mathbf{A} \cdot \dot{\mathbf{x}} - q\phi - \frac{mv^2}{2}$$



Split between parallel and perpendicular velocity

Describe **guiding center** motion $O(\epsilon^2)$

$$L = q\mathbf{A}^* \cdot \dot{\mathbf{R}} - q\phi^* - \frac{mv_{\parallel}^2}{2} + \mu \frac{m\dot{\theta}}{q}$$

Single Particle Dynamics – Second Step

Small Scales $\epsilon \sim \frac{e\phi}{T_e} \ll 1, k_{\perp}\rho_i \sim 1$

Start from

$$L = q\mathbf{A}^* \cdot \dot{\mathbf{R}} - q\phi^* - \frac{mv_{\parallel}^2}{2} + \mu \frac{m\dot{\theta}}{q}$$



Introduce small scale fluctuations

Describe **gyro center** motion $O(\epsilon^2)$

$$L = q\mathbf{A}^* \cdot \dot{\mathbf{R}} - q\phi^* - \frac{mv_{\parallel}^2}{2} + \mu \frac{m\dot{\theta}}{q} - q \langle \phi - v_{\parallel} A_{\parallel} \rangle - q \frac{\partial \langle (\phi - v_{\parallel} A_{\parallel})^2 \rangle}{\partial \mu}$$

Single Particle Dynamics – Equations of Motion

$$\dot{\mathbf{R}} = v_{\parallel} \mathbf{b} + \mathbf{v}_{\mathbf{E} \times \mathbf{B}} + \text{Other Drifts}$$



Including large scale $\frac{\nabla \phi(\mathbf{R}) \times \mathbf{B}}{B^2}$

Curvature Drift

Polarization Drift

Non-Linear Drifts

and small scale $\frac{\nabla \langle \phi(\mathbf{x}) \rangle \times \mathbf{B}}{B^2}$ fluctuations

Single Particle Dynamics – Equations of Motion

$$\dot{\mathbf{R}} = v_{\parallel} \mathbf{b} + \mathbf{v}_{\mathbf{E} \times \mathbf{B}} + \text{Other Drifts}$$

$$\dot{v}_{\parallel} = qE_{\parallel} + \mu \nabla_{\parallel} B + \text{Non-Linear Forces}$$



Including large scale $\nabla_{\parallel} \phi(\mathbf{R})$

and small scale $\nabla_{\parallel} \langle \phi(\mathbf{x}) \rangle$ fluctuations

Single Particle Dynamics – Equations of Motion

$$\dot{\mathbf{R}} = v_{\parallel} \mathbf{b} + \mathbf{v}_{\mathbf{E} \times \mathbf{B}} + \text{Other Drifts}$$

$$\dot{v}_{\parallel} = qE_{\parallel} + \mu\nabla_{\parallel}B + \text{Non-Linear Forces}$$

$$\dot{\mu} = 0$$



Conserved Adiabatic Invariant

From Single Particle to Particle Distribution

Gyrokinetic Equation

$$\frac{\partial F}{\partial t} + \dot{\mathbf{R}} \cdot \nabla F + \dot{v}_{\parallel} \frac{\partial F}{\partial v_{\parallel}} = \langle C(F) \rangle$$

Challenges

- 5-D + time
- Full Coulomb Collisions
- Coupling to Maxwell's equations (integro-differential system)

These challenges can be successfully approached by using a moment hierarchy

Our Goal – Turn Gyrokinetic Eq. Into a Hierarchy of Fluid-Like Eqs.

Expand GK Equation into a Set of 3D

Moment Hierarchy Equations

$$\frac{dn}{dt} = \dots$$

$$\frac{dv}{dt} = \dots$$

$$\frac{dT}{dt} = \dots$$



Retain necessary kinetic effects and no more

Advantages of a Moment Hierarchy Model

Set of fluid-like equations with reasonable computational cost

Tune the number of moments according to the desired level of accuracy (function of collisionality)

Reduce to the fluid model at high collisionality

3 Steps To Build a Moment Hierarchy Model

- I. Choose an orthogonal polynomial basis for F

$$F = F_M \sum_{p,j} N^{pj}(\mathbf{R}) H_p(v_{\parallel}) L_j(\mu)$$

2. Project kinetic equation onto basis

$$\int (\text{GK Eq.}) H_p L_j dv_{\parallel} d\mu$$

3. Obtain evolution equation for the basis coefficients

$$\frac{\partial N^{pj}}{\partial t} = \dots$$

From Gyrokinetic Equation to Moment Hierarchy

$$\int (\text{GK Eq.}) H_p L_j dv_{\parallel} d\mu$$



Spatial evolution of
Moments + Fields

Fluid Operator
(density, velocity,
temperature)

$$\frac{\partial N^{pj}}{\partial t} + \boxed{\nabla \cdot \dot{\mathbf{R}}^{pj}} - \boxed{\frac{\sqrt{2p}}{v_{th}} \dot{v}_{\parallel}^{p-1j}} + \boxed{\mathcal{F}^{pj}} = \boxed{C^{pj}}$$

Time Evolution

Forces included at $p>0$

Collisions

Example – 1D Linear Gyrokinetic Moment Hierarchy

Phase Mixing
(coupling with other moments)

$$\frac{\partial N^{pj}}{\partial t} + \frac{1}{2} \frac{\partial N^{p+1j}}{\partial z} + p \frac{\partial N^{p-1j}}{\partial z}$$

Time Evolution

$$= 2\delta_{p,1} K_j(k_\perp \rho) E_{\parallel} + C^{pj}$$

Electric Field Drive

Collisions

$$C^{pj}$$

Example – 1D Linear Gyrokinetic Moment Hierarchy

Phase Mixing
(coupling with other moments)

$$\frac{\partial N^{pj}}{\partial t} + \frac{1}{2} \frac{\partial N^{p+1j}}{\partial z} + p \frac{\partial N^{p-1j}}{\partial z} = 2\delta_{p,1} K_j(k_\perp \rho) E_\parallel + C^{pj}$$

Time Evolution

$$K_j(x) = \frac{1}{j!} \left(\frac{x}{2}\right)^{2j} e^{-x^2/4}$$



Analytical Closed formula for the Gyroaverage operation

Full finite Larmor radius effects

Projection of the Full Coulomb Collision Operator

$$C^{pj} = \int \langle C(F) \rangle H_p L_j d\mathbf{v}_{\parallel} d\mu$$

with $C(F) = \frac{\partial}{\partial \mathbf{v}} [\mathbf{A}(F)F] + \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : [\mathbf{D}(F)F]$

- Bilinear
- Tensorial Nature
- Gyroaveraging Operation
- No parallel/perpendicular velocity symmetries

Not immediate...

Collision Operator - Spherical Harmonic Decomposition

$$C[F] \sim C[\mathbf{P}^{lk}(\mathbf{v})] \sim \mathbf{P}^{lk}(\mathbf{v})$$

Gyroaveraging Procedure

Large Scales $k_{\perp}\rho_i \ll 1$

$$\langle \mathbf{P}^{lk}(\mathbf{v}) \rangle \quad \xrightarrow{\hspace{1cm}} \quad H_p(v_{\parallel})L_j(\mu)$$

Small Scales $\frac{e\phi}{T_e} \ll 1$

$$\langle \mathbf{P}^{lk}(\mathbf{v})e^{in\theta} \rangle \quad \xrightarrow{\hspace{1cm}} \quad f(n)H_p(v_{\parallel})K_j(k_{\perp}\rho)L_j(\mu)$$

Moments of the Collisional Operator

$$C^{pj} = \int \langle C(F) \rangle H_p L_j dv_{\parallel} d\mu$$

After integration



$$C^{pj} = \text{Kernel} \times f(n) \times \text{moments of F}$$

Physics That We Are Able To Capture

Low Collisionality



Particle resonance effects
retrieved

High-Collisionality
Semi-collisional closure

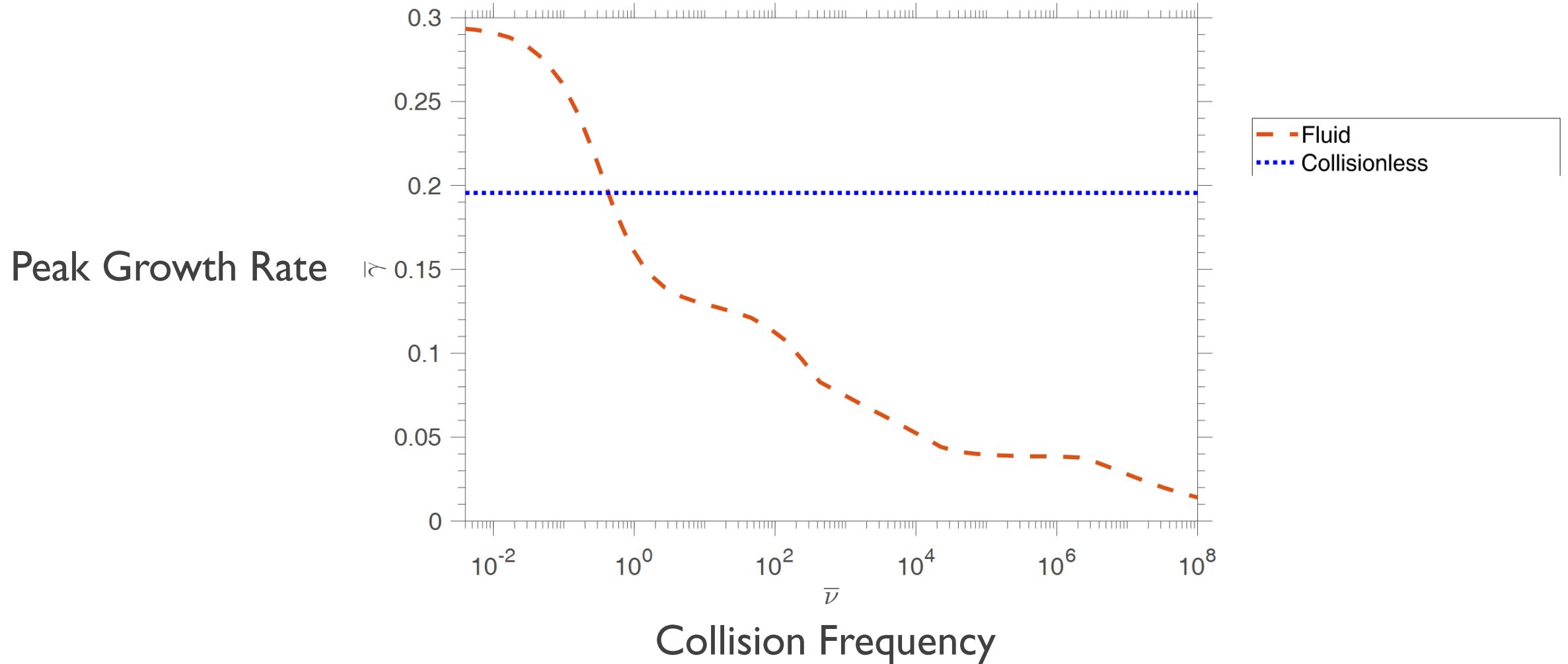
$$\frac{\delta F}{F_M} \sim \frac{\lambda_{\text{mfp}}}{L_{\parallel}}$$



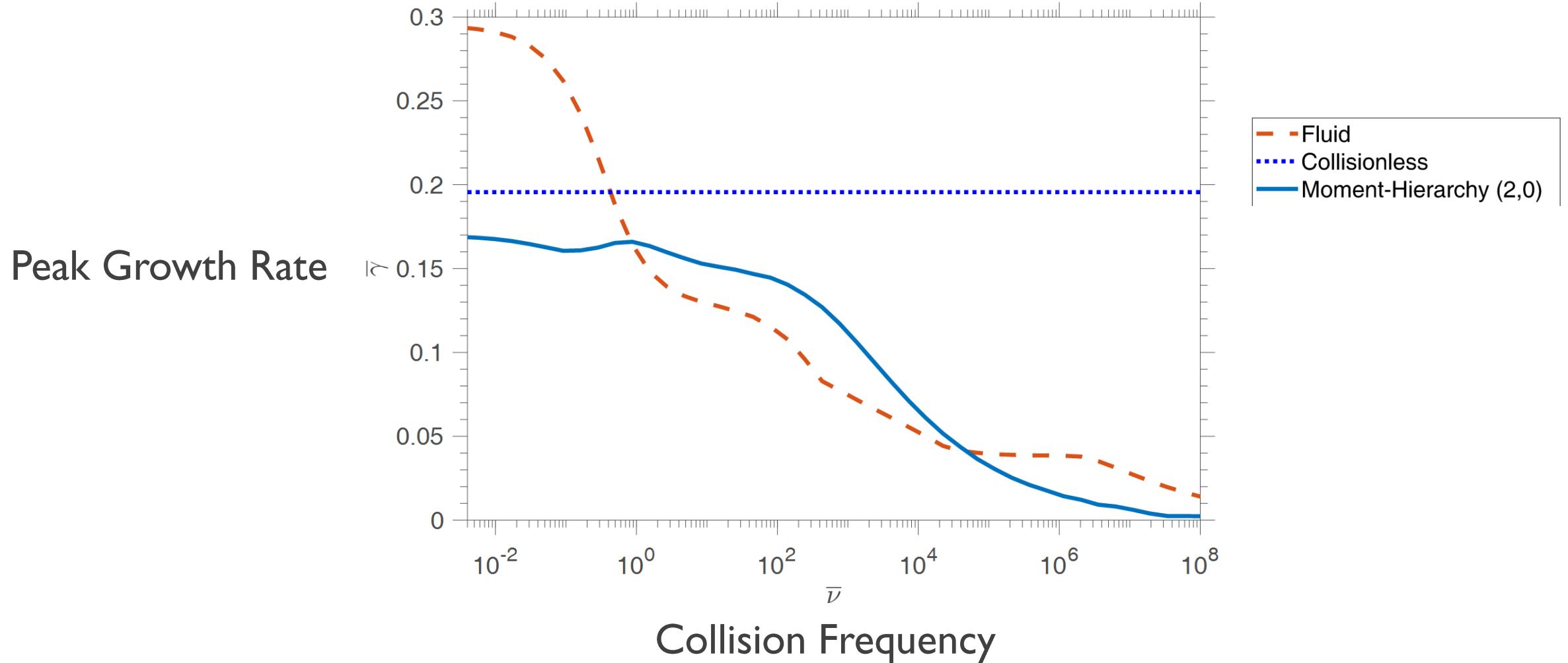
Drift-Reduced Braginskii
equations retrieved

$$N^{30} \sim q_{\parallel} = -\chi_{\parallel} \nabla_{\parallel} T$$

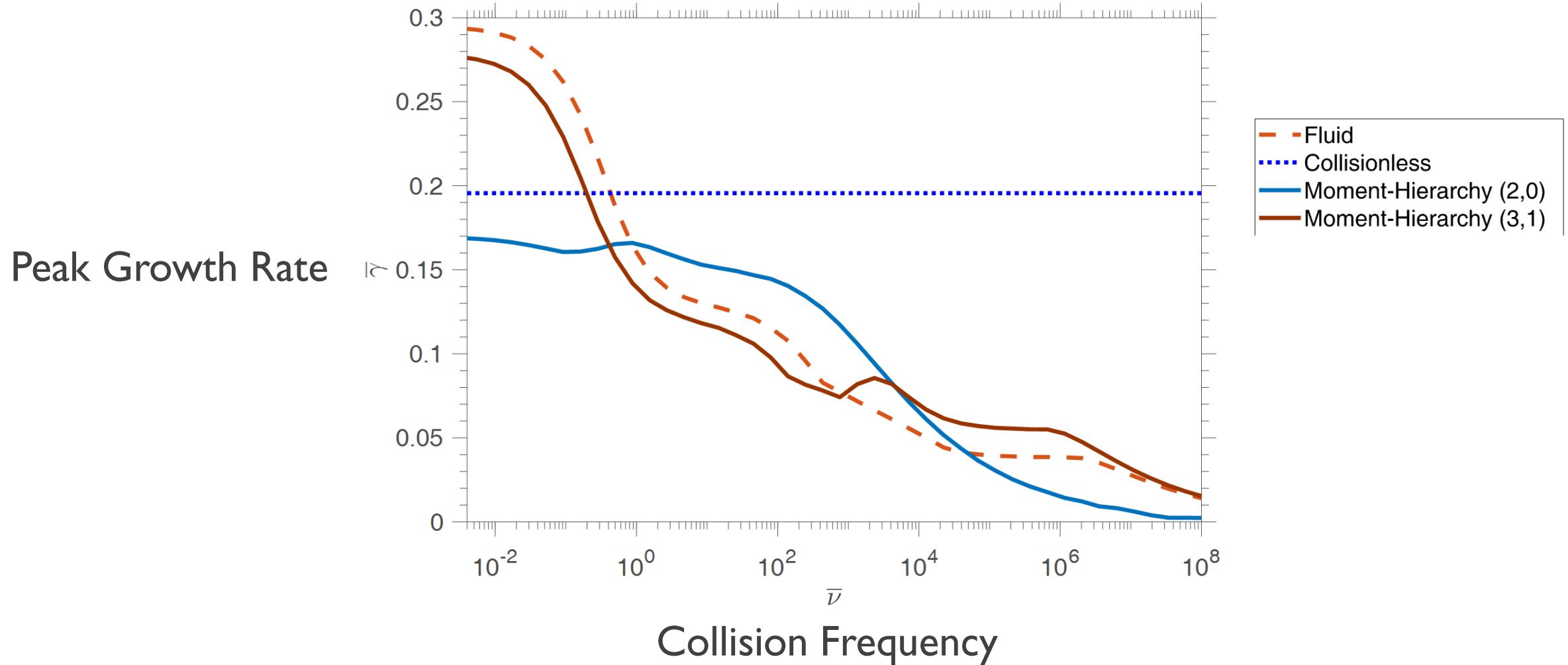
Drift-Wave Instability Case Study



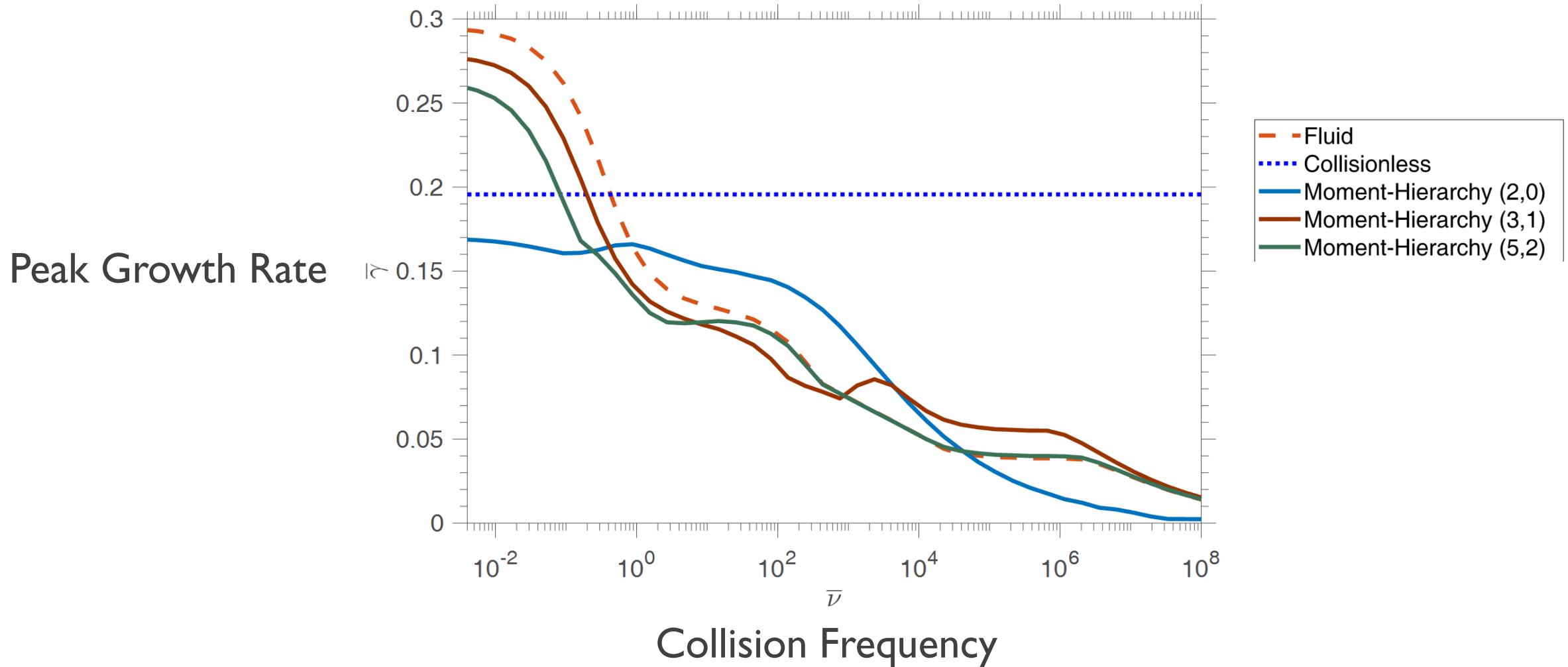
Drift-Wave Instability Case Study



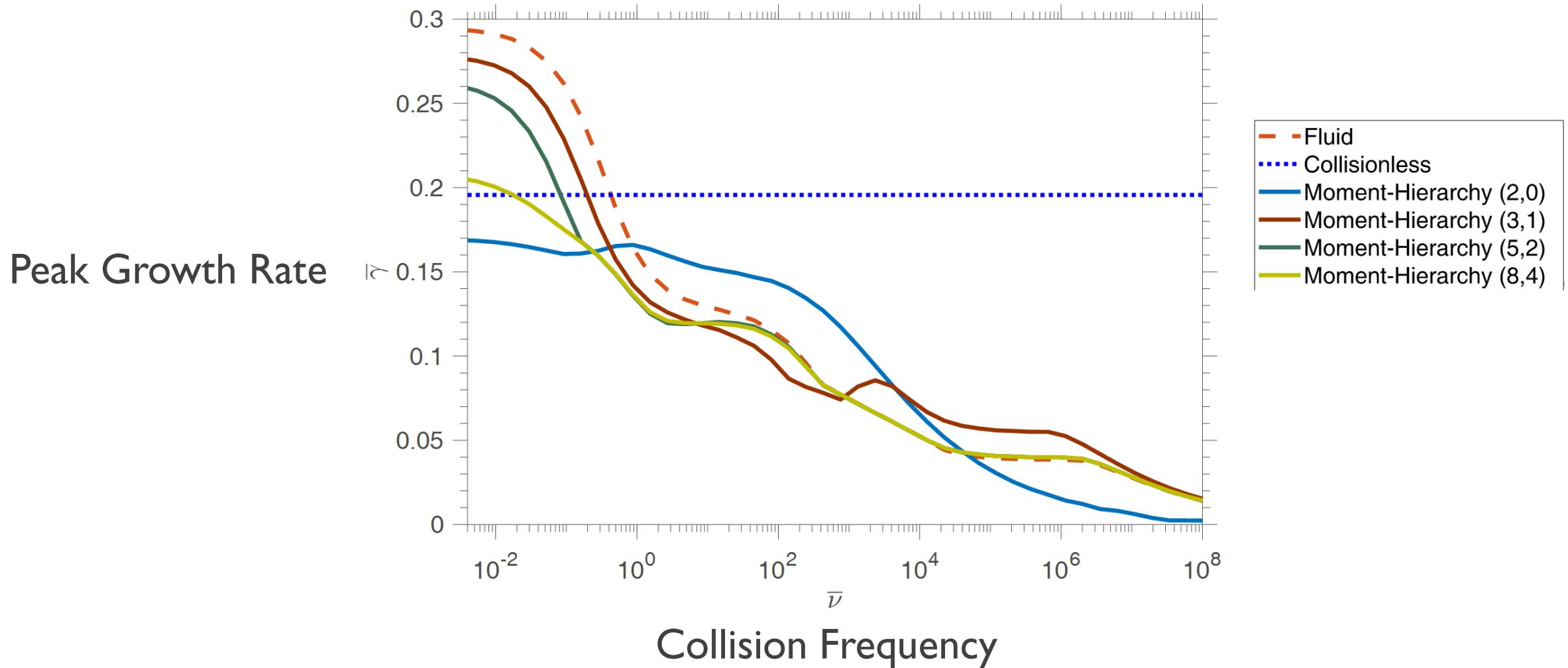
Drift-Wave Instability Case Study



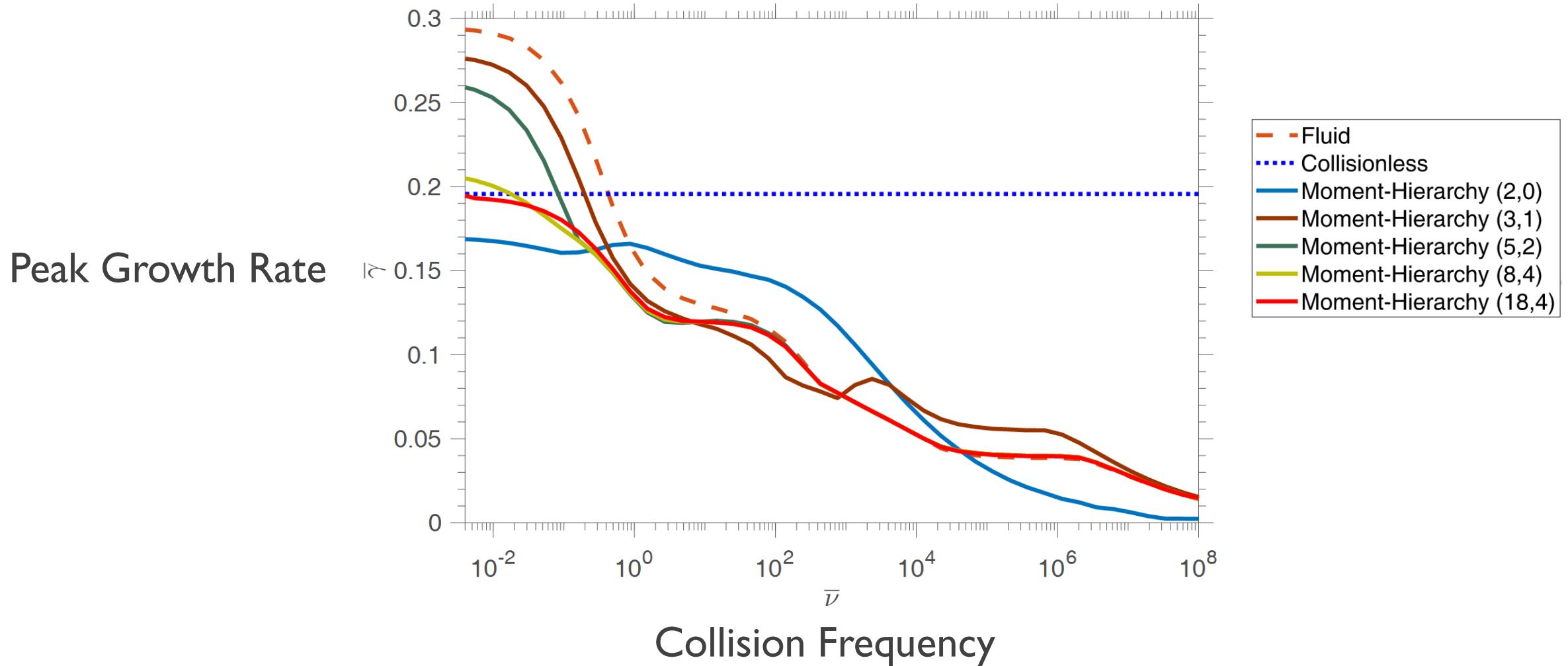
Drift-Wave Instability Case Study



Drift-Wave Instability Case Study



Drift-Wave Instability Case Study



Summary

	This Work	Qin et al. 2006 ¹	Hahm et al. 2009 ²	Dimitis et al. 2012 ³	Madsen et al. 2013 ⁴	Mandell et al. 2018 ⁵
Large Scales	EM $O(\epsilon^2)$	EM $O(\epsilon)$	ES $O(\epsilon^2)$	ES $O(\epsilon^2)$	ES $O(\epsilon)$	ES $O(\epsilon)$
Small Scales	EM $O(\epsilon^2)$	EM $O(\epsilon^2)$	EM $O(\epsilon^2)$	EM $O(\epsilon^2)$	EM $O(\epsilon)$	ES $O(\epsilon)$
Collisions	Yes	No	No	No	No	No
Poisson's Eq.	Moments	$\int(\dots) d^3v$	Long-Wavelength Limit	$\int(\dots) d^3v$	Long-Wavelength Limit, Padé	Moments
B_{\parallel}^*	Exact	Exact	Exact	Exact	$O(\epsilon)$	B

¹ Qin et al., Contrib. Plasma Phys. **46** (2006)

² Hahm et al., Physics of Plasmas **16** (2009)

³ Dimitis, Physics of Plasmas **19** (2012)

⁴ Madsen, Physics of Plasmas **20** (2013)

⁵ Mandell et al., J. Plasma Phys **84** (2018)

Acknowledgements

This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014-2018 under grant agreement No 633053.

RJ was partially funded by the Portuguese FCT - Fundação para a Ciência e Tecnologia, under grant PD/BD/105979/2014.

NFL was partially funded by US Department of Energy Grant no. DE-FG02-91ER54109.

The views and opinions expressed herein do not necessarily reflect those of the European Commission.