Regimes of tearing modes with parallel dynamics having real frequencies

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Reduced MHD formalism with parallel dynamics elucidates origin of the Glasser Effect

In this poster we show a streamlined derivation of the Glasser effect in the resistive-inertial (RI) and visco-resistive (VR) tearing regimes. These calculations include parallel dynamics but neglect both the divergence of the $E \times B$ drift $\nabla \cdot \mathbf{v}_\perp$, and perpendicular resistivity $\eta_\perp$. The purpose of this derivation is threefold:

(1) to show that these two last effects are not necessary to obtain the qualitative results, i.e. complex roots and stabilization for positive constant-ψ matching parameter Δ',

(2) to illustrate this simple approach for use in other tearing regimes,

(3) to exploit the simplicity of this model to elucidate the physics.

Applications of finite frequency tearing layers: EF locking and RWM

- We recently demonstrated [1] that the Maxwell torque on the plasma in the presence of an applied error field is modified significantly for tearing modes having real frequencies near marginal stability.
- Finally, we find that the lowering of the threshold for destabilization of the resistive wall mode can be much more pronounced than observed for tearing modes in Ref. [2].
- In all regimes studied in this poster, the existence of finite frequency tearing modes in the plasma frame is related to nearby electrostatic resistive interchange modes with complex frequencies.

References: [1] J. M. Finn, A. J. Cole, and D. P. Brennan, PoP (Letters) 22, 120701 (2015). [2] J. M. Finn and R. A. Gerwin, PoP 3, 2344 (1996)

Review of TModes and RI regime with parallel dynamics -I

Spontaneous modes

- TM are 'slow' growing: obey marginally stable (no inertia or viscosity) ideal MHD everywhere except near boundary layer at $k \cdot B_0 = 0$
- Dispersion relation from asymptotic matching logarithmic derivative in flux function across tearing layer

$$\Delta' = \Delta(\gamma - i\omega_r)$$

$$\rho\gamma\nabla_\perp^2\tilde{\phi} = iF\nabla_\perp^2\tilde{\psi} + \frac{2imB_\theta^2}{B_0^2 r^2}\tilde{p}, \qquad (1)$$

$$\gamma\tilde{\psi} = iF\tilde{\phi} + \eta\nabla_\perp^2\tilde{\psi}, \quad \text{(a)} \qquad (2)$$

$$\tilde{p} = -\frac{imp'}{r}\frac{1}{\gamma^2 + k_{||}^2 c_s^2}\left(\gamma\tilde{\phi} - \frac{ik_{||}c_s^2}{B_0}\tilde{\psi}_0\right). \qquad (3)$$

RI regime with parallel dynamics -II

Sound speed parameter

$$b^2 = \alpha^2 \delta^2 c_s^2 / \gamma_{ri}^2 B_0^2 = b_0^2 Q^{1/2} \tag{4}$$

Normalized Suydam (Mercier in toroidal) parameter

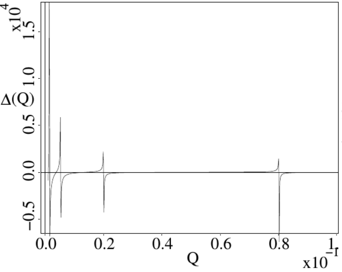
$$G = -\frac{2m^2 B_\theta^2(r_t) p'(r_t)}{B_0^2 r_t^3} \frac{\delta^2}{\rho \gamma_{ri}^2} \frac{1}{Q^{3/2}} = \frac{G_0}{Q^{3/2}}, \tag{5}$$

Layer equation for normalized stream function

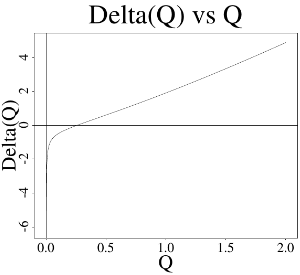
$$\frac{d^2 W}{d\xi^2} - \xi^2 W + \frac{Q^2}{Q^2 + b^2 \xi^2} G W = -\left(1 + \frac{G b^2}{Q^2 + b^2 \xi^2}\right) \xi. \tag{6}$$

RI regime with parallel dynamics - Results I

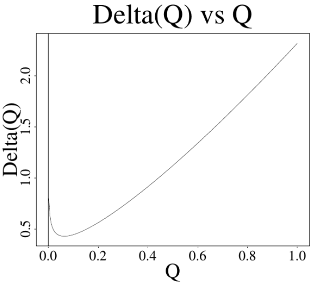
- Non monotonic behavior of tearing layer matching parameter (leading to complex roots) caused by presence of poles related to electrostatic resistive interchange modes.
- Increasing sound speed (b_0) leads to coalescence of poles at origin



$b = 0.1$, $G_0 = 0.1$, $G_0/b_0^2 = 10$



$G_0/b_0^2 = 0.055$



$G_0/b_0^2 = -0.055$

VR regime with parallel dynamics -I

We replace Eq. (1) with

$$0 = iF\nabla_\perp^2 \tilde{\psi} + \frac{2imB_\theta^2}{B_0^2 r^2}\tilde{p} + \mu\nabla_\perp^2 \tilde{\phi}, \tag{7}$$

as in the RI regime $E = -2m^2 B_\theta^2 p'/\gamma B_0^2 r_t^3 = E_0/\gamma$. As usual in the VR regime we find $\delta^6 = \eta\mu/\alpha^2$.

$$G = \frac{G_0}{Q}. \tag{8}$$

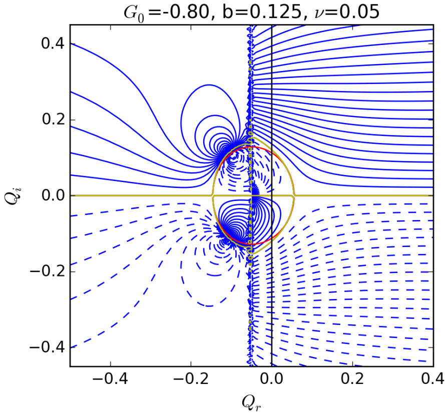
VR regime with parallel dynamics -II

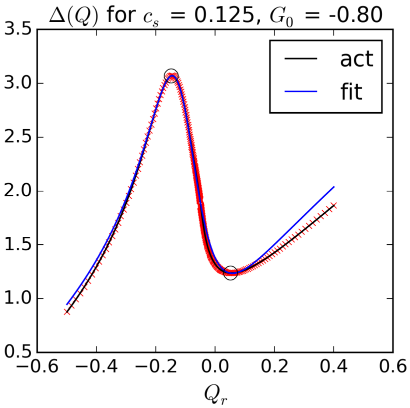
$$\frac{d^4W}{d\xi^4} + \xi^2 W - \frac{GQ^2}{Q^2 + b^2\xi^2} W = \left(1 + \frac{Gb^2}{Q^2 + b^2\xi^2}\right)\xi, \qquad (9)$$

For unfavorable curvature ($G_0 > 0$), the symmetry $G_0 \to -G_0$, $Q \to -Q$ shows that similar poles occur, but they correspond to growing modes when the roots for $G_0 < 0$ are damped.

VR regime with parallel dynamics -Results I

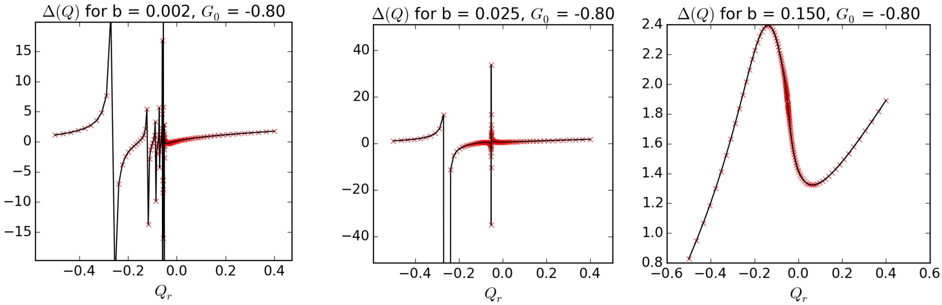
Locus of roots, real frequency at marginal stability, and $\Delta'_c > 0$ (Glasser Effect) in viscous tearing regime. $Q \rightarrow Q + Q_0$ to simulate perpendicular resistivity.
Non monotonic just as in GGJ-RI.





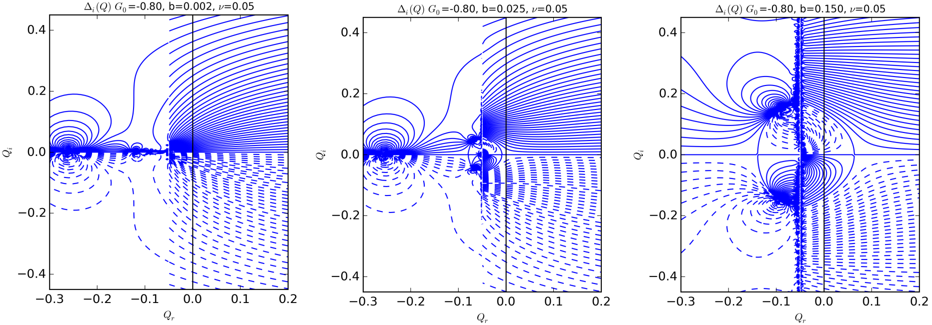
VR regime with parallel dynamics -Results II

Non monotonic behavior of tearing layer matching parameter (leading to complex roots) caused by poles of resistive interchange modes



VR regime with parallel dynamics -Results III

Resistive interchange poles move into sound wave continuum as sound speed increases.



Review of driven tearing modes and net Maxwell Torque on plasma

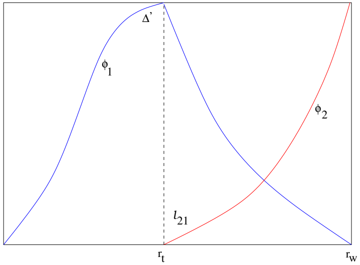
EF problem: driven response in stable plasma

- Consider static single harmonic (k) 3D field resonant on single surface; Replace $\gamma - i\omega_r \rightarrow ikV$
- EF induces net electromagnetic (Maxwell) force only on resonant surface

$$F_m \propto -|\tilde{\psi}_t|^2 \,\mathrm{Im}\left[\Delta(ikV)\right] \;\ldots\; \tilde{\psi}_t \propto \frac{\tilde{\psi}_w}{\Delta' - \Delta(ikV)},$$

  with $\tilde{\psi}_w$ the error field strength at vessel wall, etc
- Model of EF locking: anomal. $\mu_\perp$ resists F_m: $F_m + F_\mu = 0 \rightarrow$ leads to bifurcation, hysteresis

Calculating the reconnected flux from the error field



$$\tilde{\psi}(r) = \alpha_1 \phi_1(r) + \alpha_2 \phi_2(r)$$
$$[\tilde{\psi}']_{r_t} = \Delta(\gamma_d)\tilde{\psi}(r_t) \qquad \gamma_d = \gamma + ikv$$
$$\alpha_1 \Delta' + l_{21}\alpha_2 = \Delta(\gamma_d)\alpha_1 \qquad \alpha_2 = \tilde{\psi}(r_w)$$

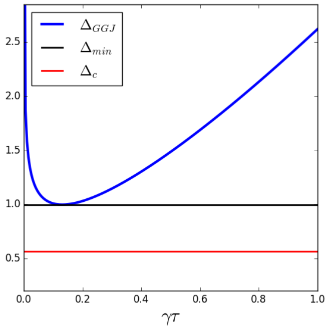
Steady-state: $\gamma_d = \gamma + ikv \to ikv$

$$\tilde{\psi}(r_t) = \alpha_1 = -\frac{l_{21}\tilde{\psi}(r_w)}{\Delta' - \Delta(ikv)}$$

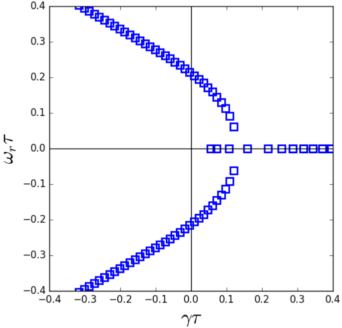
Dispersion relation in the denominator, as usual.

RI-GGJ: a familiar regime with finite frequency modes

$$L_r \Delta' = 2.12 \left[(\gamma\tau - i\tau\omega_r)^{5/4} - \frac{\pi D}{4(\gamma\tau - i\tau\omega_r)^{1/4}} \right] \quad \ldots\ D \sim -p'(1-q^2) < 0$$



$\Delta(\gamma\tau)$ vs. $\gamma\tau$



Locus of roots for RI with fixed $D < 0$, varying Δ'

complex roots if $L_r \Delta' < \Delta_{min}$,

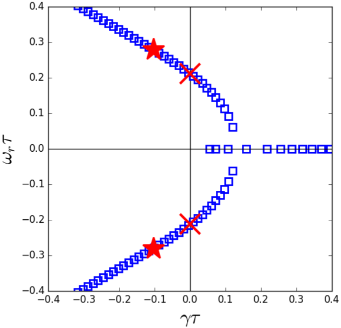
stabilized if $L_r \Delta' < \Delta_c$

$L_r \propto S^{-1/3}$ and $\tau \sim S^{1/3}$

Driven EF problem sweeps along imaginary axis, distance to poles in $\tilde{\psi}_t$ influences force amplitude

$$F_m \propto -|\tilde{\psi}_t|^2 \,\mathrm{Im}\left[\Delta(ikV)\right] \;\ldots\; \tilde{\psi}_t \propto \frac{\tilde{\psi}_w}{\Delta' - \Delta(ikV)}$$

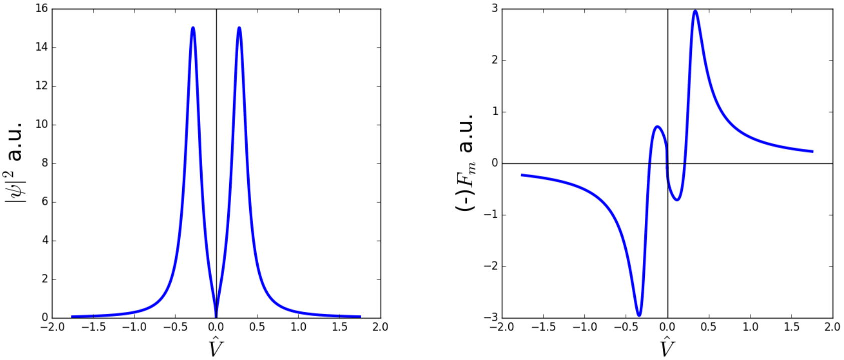
- Typical of inhomogeneous solutions: dispersion relation in denominator
- Zeros of $\mathrm{Im}\left[\Delta(ikV)\right]$ at locus crossings (and $V = 0$)
- Plasma conditions determine appropriate Δ, layer regime
- Note: two-fluid regimes lack symmetry in $\pm\omega_r$, $\omega_{*i,e}$



EF Maxwell force zero and largest magnetic flux observed when surface flows at rate $V \sim \omega_r / k$—-RI results

$$F_m \propto -|\tilde{\psi}_t|^2 \,\mathrm{Im}[\Delta(ikV)] \sim -\frac{\Delta_i(ikV\tau)}{(L_r\Delta' - \Delta_r(ikV\tau))^2 + \Delta_i(ikV\tau)^2}$$

Numerator = 0 where $\Delta_i = 0$; denominator minimum nearby, both $\omega_r \approx kV$



- New: pronounced peaks in $|\psi(r_t)|^2$ off axis, $L_r\Delta' = 0.25$
- New: zero of F_m at finite value of $\hat{V}$!

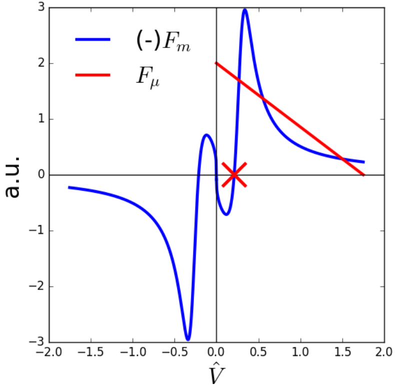
Main result: Fields are locked to static error field, plasma flow locked to finite value $V \gtrsim \omega_r/k$—RI regime

Steady state force balance $F_m + F_\mu = 0$, with $F_\mu \propto \mu(V_0 - V)$ viscous force across layer and

$$F_m \propto -\frac{\Delta_i(ikV\tau)}{(L_r\Delta' - \Delta_r(ikV\tau))^2 + \Delta_i(ikV\tau)^2}$$

Different ways to induce bifurcation. Decreasing μ equiv. to increasing $\tilde{\psi}_t$

- Large μ: 3 roots (2 stable, 1 unstable)
- Intermediate μ intersects at $V \gtrsim \omega_r/k$.
- **Driven B field** locked to static EF
- **Flow** locked to $V \gtrsim \omega_r/k$; not $V \gtrsim 0$. Asymptote $V \nrightarrow 0$.
- For very small μ two other states are possible.



EF Maxwell force zero VR regime

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- Intermediate μ intersects at $V \gtrsim \omega_r/k$.
- **Driven B field** locked to static EF
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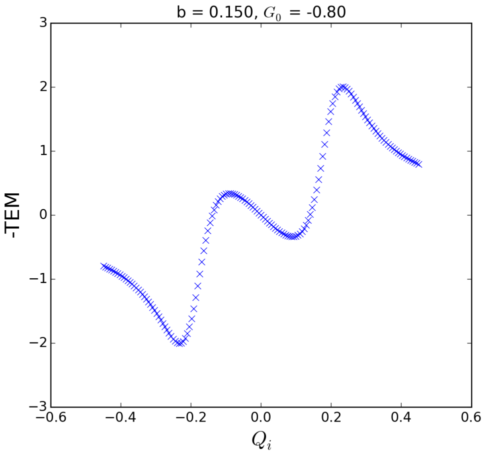
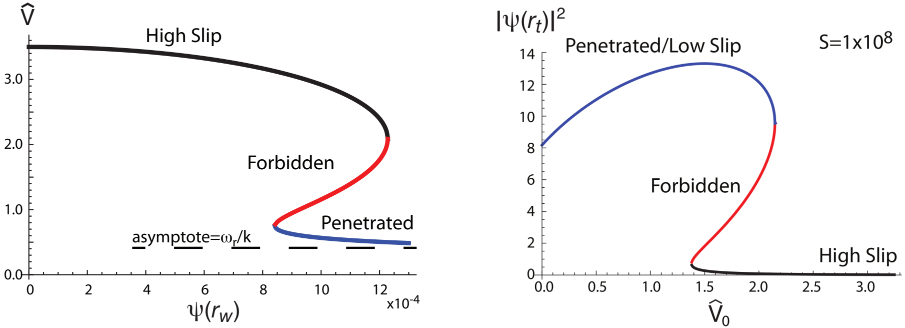


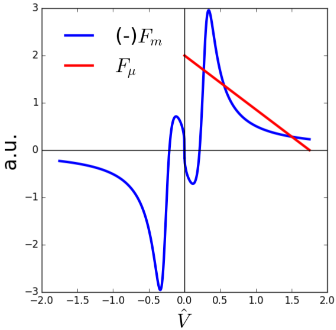
EF force balance exhibits bifurcation to a high reconnected flux, low flow “locked” state—RI Regime

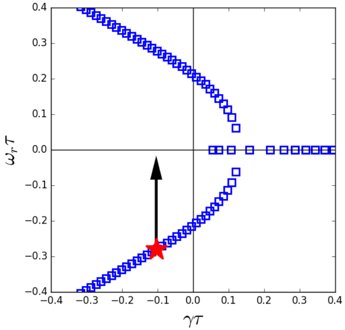
- Two different aspects of the same bifurcation behavior



- Fixed initial velocity $\hat{V}_0$, increasing $\tilde{\psi}(r_w)$
- Fixed $\tilde{\psi}(r_w)$, decreasing $\hat{V}_0$

New observation: “locked” state corresponds to backward propagating wave in plasma frame $V \sim \omega_r/k$





- Locked state occurs when Doppler shifted $(+kV)$ *driven mode* interacts most with zero frequency error field.
- Asymptotes to finite value $V \to \omega_r/k$ (as $\gamma, \mu \to 0$ -typ. damped-driven) .
- Locked state has a backward propagating wave in the plasma frame.

Riccati equation for shooting with stiff equations

Homogeneous form of RI equation for streamfunction

$$W'' - x^2 W = 0$$

Basic Riccati: let $\eta = W'/W$. Lie symmetry: $W \to \lambda W$.
Obtain reduction to first order for η:

$$\eta' = x^2 - \eta^2$$

As $x \to +\infty$, $\eta \to \pm x$ (WKB)

$$\eta = \pm x - \frac{1}{2x} \quad \left(W \approx x^{-1/2} e^{\pm x^2/2} \right)$$

Shooting with BC $W \to e^{-x^2/2}$ as $x \to \infty$: Solution with $\eta = -x$ has $\delta\eta \sim e^{x^2}$ – unstable for shooting forward. Stable backwards. ($\eta = x$ has $\delta\eta \sim e^{-x^2}$.)

Riccati equation

$$W = Ae^{x^2/2} + Be^{-x^2/2} \quad \text{large } x$$

$$\eta = x\frac{Ae^{x^2/2} - Be^{-x^2/2}}{Ae^{x^2/2} + Be^{-x^2/2}}$$

$$= x\frac{A - Be^{-x^2}}{A + Be^{-x^2}}$$

For $W \sim e^{-x^2/2}$ integrate backwards. Also, easily formulated as second order system

$$W' = Z, \;\; Z' = x^2 W \quad \eta = Z/W$$

Vector Riccati equation, n'th order

$$\phi_i'(x) = \sum_{j=1}^{n} a_{ij}\phi_j(x)$$

Same Lie symmetry, $\eta_i = \phi_i/\phi_n$

$$\eta_i' = \frac{\phi_i'}{\phi_n} - \frac{\phi_i\phi_n'}{\phi_n^2}$$

$$= a_{ij}\eta_j - a_{nj}\eta_i\eta_j$$

$i = n$ trivial ($\eta_n = 1$) reduction of order by one.

Fourth order example:

$$\phi_1(x) = A_1 e^{\alpha_1 x^{p_1}} + A_2 e^{\alpha_2 x^{p_2}} + A_3 e^{\alpha_3 x^{p_3}} + A_4 e^{\alpha_4 x^{p_4}}$$

Still works to take most stable solution and integrate η_i backwards.

Application to tearing layers – resistive MHD with parallel dynamics

. Equations for vorticity, flux, pressure, parallel velocity. Four equations, each with a $k_{||} = k'_{||} x$. No constant ψ approx.
. Fourier transform $x \to k \implies$ fourth order system.
$k_{||} \to k'_{||} \partial/\partial k$
. Resistive inertial (RI) and viscoresistive (VR): same order in k. Inclusion of classical diffusion $\eta_\perp \tilde{p}''(x) \to -\eta_\perp k^2 \hat{p}(k)$ ... no change in order but removes the parallel dynamics continuum.
. Integrate backwards in k using Riccati to get Δ' matching condition. $\tilde{\phi} \to x^{p_1}, \ x^{p_2}$ as $x \to \infty \implies \hat{\phi}(k) \to k^{-p_1-1}, \ k^{-p_2-1}$ as $k \to 0$
. Preliminary results: agreement with constant- ψ results for small γ