# General field theory and weak Euler-Lagrange equation for classical particle-field systems 

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#### Abstract

A general field theory for classical particle-field systems is developed. Compared with the standard classical field theory, the distinguish feature of a classical particle-field system is that the particles and fields reside on different manifolds. The fields are defined on the 4D space-time, whereas each particle's trajectory is defined on the 1D time-axis. As a consequence, the standard Noether's procedure for deriving local conservation laws in space-time from symmetries is not applicable without modification. To overcome this difficulty, a weak Euler-Lagrange equation for particles is developed on the 4D space-time, which plays a pivotal role in establishing the connections between symmetries and local conservation laws in space-time. Especially, the non-vanishing Euler derivative in the weak Euler-Lagrangian equation generates a new current in the conservation laws. Several examples from plasma physics are studied as special cases of the general field theory. In particular, the relations between the rotational symmetry and angular momentum conservation for the Klimontovich-Poisson system and the Klimontovich-Darwin system are established.


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## I. INTRODUCTION

It has been widely accepted as a fundamental principle of physics that conservation laws of particle systems or field systems can be derived from the symmetries that the systems admit. This is the well-known Noether's theorem [1].

Classical particle-field systems, where many particles evolve under self-generated interacting fields, are often encountered in plasma physics [2-11], astrophysics [12 18], and accelerator physics [19, 20]. For classical systems with particles and self-generated interacting fields, the connections between conservation laws and symmetries have been established only recently [21 23]. It was pointed out [22, 23] that the standard Euler-Lagrange (EL) equation for particles are not applicable in Noether's procedure, because the dynamics of particles and fields are defined on manifolds with different dimensions. Instead, a weak EL equation for particles should be used to establish the link between the conservation laws and symmetries.

The systems discussed in [22, 23] are some special particle-field systems such as the Klimontovich-Poisson (KP) system, the Klimontovich-Darwin (KD) system and the KlimontovichMaxwell (KM) system. And only a special symmetry, i.e., the space-time translation symmetry, is considered. In this study, we extend the theory to general symmetries in general particle-field systems. The generalized theory can be also viewed as a generalized version of Noether's theorem for systems with classical particles and fields residing on different manifolds.

As special cases and applications of the general theory, we study the time translation symmetry of the KP system and the rotational symmetry for the KP and KD systems. The energy conservation law of the KP system, as a result of the time translation symmetry, agrees with the result of Ref. [22]. The relations between the rotational symmetry and angular momentum conservation for the KP and KD systems are established. In this case, the rotation of the vector potential for the KD system needs to be included as a part of the symmetry that the system admits. Without the rotation of the vector potential, the rotation of the position alone does not preserve the Lagrangian. Of course, the rotation of the vector potential is the representation of the rotational symmetry in the fiber of the vector bundle at each space-time location.

This paper is organized as follows. In Sec.II, we introduce the action of a general particle-
field system. The weak EL equation is developed as necessitated by the fact that classical particles and fields live on different manifolds. Symmetries for the system are discussed in Sec.III, and the links between conservation laws and symmetries are established. Special symmetries and conservation laws for the KP and KD systems are derived in Secs.IV and V.

## II. GENERAL CLASSICAL PARTICLE-FIELD SYSTEMS AND WEAK EULERLAGRANGE EQUATION

In general, the action of a classical particle-field system is

$$
\begin{equation*}
\mathcal{A}=\sum_{a} \int L_{a}\left(t, \boldsymbol{X}_{a}(t), \dot{\boldsymbol{X}}_{a}(t), \boldsymbol{\psi}\left(t, \boldsymbol{X}_{a}(t)\right)\right) d t+\int \mathcal{L}_{F}\left(t, \boldsymbol{x}, \boldsymbol{\psi}, \frac{\partial \boldsymbol{\psi}}{\partial t}, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}}\right) d t d^{3} \boldsymbol{x} \tag{1}
\end{equation*}
$$

where $\boldsymbol{X}_{a}(t)$ is the trajectory of the $a$-th particle and $\boldsymbol{\psi}=\boldsymbol{\psi}(t, \boldsymbol{x})$ is a field of scalar, vector, or tensor type. Apparently, the dynamics of particles and fields are defined on different manifolds. The field $\boldsymbol{\psi}$ is on the 4D space-time, whereas each particle's trajectory is on the 1D time-axis. Thus, the integral of the Lagrangian density $\mathcal{L}_{F}$ for the field $\boldsymbol{\psi}$ is over space-time, and the integral of Lagrangian $L_{a}$ for the $a$-th particle is over time only.

Because of this fact, the action defined in Eq. (1) is not easily applicable to Noether's procedure of derving conservation laws in space-time. To overcome this difficult, we multiply the first part in the right-side of Eq. (1) by the identity

$$
\begin{equation*}
\int \delta_{a} d^{3} \boldsymbol{x}=1 \tag{2}
\end{equation*}
$$

where $\delta_{a} \equiv \delta\left(\boldsymbol{x}-\boldsymbol{X}_{a}(t)\right)$ is the Dirac $\delta$-function. The action $\mathcal{A}$ in Eq. (1) is then transformed into an integral over space-time,

$$
\begin{equation*}
\mathcal{A}=\int \mathcal{L}\left(t, \boldsymbol{x}, \boldsymbol{X}_{a}(t), \dot{\boldsymbol{X}}_{a}(t), \boldsymbol{\psi}, \frac{\partial \boldsymbol{\psi}}{\partial t}, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}}\right) d t d^{3} \boldsymbol{x} \tag{3}
\end{equation*}
$$

where $\mathcal{L}$ is the Lagrangian density of the particle-field system defined as

$$
\begin{equation*}
\mathcal{L}=\sum_{a} \mathcal{L}_{a}+\mathcal{L}_{F}, \mathcal{L}_{a}=L_{a}\left(t, \boldsymbol{X}_{a}(t), \dot{\boldsymbol{X}}_{a}(t), \boldsymbol{\psi}(t, \boldsymbol{x})\right) \delta_{a} . \tag{4}
\end{equation*}
$$

Now we determine how the action given by Eq. (3) varies in response to the variation of $\psi$,

$$
\begin{equation*}
\delta \mathcal{A}=\int \boldsymbol{E}_{\psi}(\mathcal{L}) \cdot \delta \boldsymbol{\psi} d t d^{3} \boldsymbol{x} \tag{5}
\end{equation*}
$$

where the symbol "." stands for total contraction between two tensors, and $\boldsymbol{E}_{\psi}$ denotes the Euler operator

$$
\begin{equation*}
\boldsymbol{E}_{\psi}(\mathcal{L}) \equiv \frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi}}-\frac{D}{D t}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)}\right]-\frac{D}{D \boldsymbol{x}} \cdot\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial \boldsymbol{x}}\right)}\right] \tag{6}
\end{equation*}
$$

Applying Hamilton's principle to Eq. (5), we immediately obtain the equation of motion for the field,

$$
\begin{equation*}
\boldsymbol{E}_{\psi}(\mathcal{L})=0 \tag{7}
\end{equation*}
$$

by the arbitrariness of $\delta \boldsymbol{\psi}$.
Next, we derive the equation of motion for particles. There are two ways to proceed. If we start from the action defined in Eq. (1), the variation of $\mathcal{A}$ induced by $\delta \boldsymbol{X}_{a}$ is

$$
\begin{equation*}
\delta \mathcal{A}=\sum_{a} \int\left[\frac{\partial L_{a}}{\partial \boldsymbol{X}_{a}}-\frac{D}{D t}\left(\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right)\right] \cdot \delta \boldsymbol{X}_{a} d t \tag{8}
\end{equation*}
$$

and the EL equation of the $a$-th particle is

$$
\begin{equation*}
\frac{\partial L_{a}}{\partial \boldsymbol{X}_{a}}-\frac{D}{D t}\left(\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right)=0 \tag{9}
\end{equation*}
$$

Since Eq. (9) is not a differential equation on space-time, it cannot be directly adopted in Noether's procedure of deriving conservation laws.

The alternative way is to use the action defined in Eq. (3), which varies as

$$
\begin{equation*}
\delta \mathcal{A}=\sum_{a} \int d t \delta \boldsymbol{X}_{a} \cdot\left[\int\left[\frac{\partial \mathcal{L}}{\partial \boldsymbol{X}_{a}}-\frac{D}{D t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}}\right)\right] d^{3} \boldsymbol{x}\right] \tag{10}
\end{equation*}
$$

in response to the variation of $\boldsymbol{X}_{a}$. Here, the term $\delta \boldsymbol{X}_{a}$ in Eq. (10) was moved outside from the integral $\int[\cdots] d^{3} \boldsymbol{x}$ because it is independent of $\boldsymbol{x}$. Hamilton's principle, i.e., $\delta \mathcal{A}=0$ for the variation $\delta \boldsymbol{X}_{a}$, requires the integral over the configuration space vanishes,

$$
\begin{equation*}
\int \boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L}) d^{3} \boldsymbol{x}=0 \tag{11}
\end{equation*}
$$

Here, $\boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})$ is the Euler operator with respect to $\boldsymbol{X}_{a}$,

$$
\begin{equation*}
\boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L}) \equiv \frac{\partial \mathcal{L}}{\partial \boldsymbol{X}_{a}}-\frac{D}{D t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}}\right) \tag{12}
\end{equation*}
$$

Following Refs. [22, 23], Eq. (11) is called submanifold EL equation because it is defined only on the time-axis after integrating over the spatial dimensions. Both Eqs. (9) and (11) describe the equation of motion of the $a$-th particle. The equivalence of the two equations can be easily proved as follows,

$$
\begin{align*}
& \frac{\partial L_{a}}{\partial \boldsymbol{X}_{a}}-\frac{D}{D t}\left(\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right)=\frac{\partial}{\partial \boldsymbol{X}_{a}}\left(\int \mathcal{L}_{a} d^{3} \boldsymbol{x}\right)+\frac{D}{D t}\left[\frac{\partial}{\partial \dot{\boldsymbol{X}}_{a}}\left(\int \mathcal{L}_{a} d^{3} \boldsymbol{x}\right)\right] \\
& =\int\left[\frac{\partial \mathcal{L}_{a}}{\partial \boldsymbol{X}_{a}}-\frac{D}{D t}\left(\frac{\partial \mathcal{L}_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right)\right] d^{3} \boldsymbol{x}=\int \boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L}) d^{3} \boldsymbol{x} . \tag{13}
\end{align*}
$$

In Eq. (11), the vanishing integral over the configuration space suggests that the integrand $\boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})$ could be a total divergence. We now derive such an explicit expression for it. For the first term in $\boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})$,

$$
\begin{align*}
\frac{\partial \mathcal{L}}{\partial \boldsymbol{X}_{a}} & =\frac{\partial}{\partial \boldsymbol{X}_{a}}\left(L_{a} \delta_{a}\right)=L_{a} \frac{\partial \delta_{a}}{\partial \boldsymbol{X}_{a}}+\frac{\partial L_{a}}{\partial \boldsymbol{X}_{a}} \delta_{a} \\
& =-L_{a} \frac{D \delta_{a}}{D \boldsymbol{x}}+\frac{\partial L_{a}}{\partial \boldsymbol{X}_{a}} \delta_{a}=\frac{D}{D \boldsymbol{x}} \cdot\left(-\mathcal{L}_{a} \boldsymbol{I}\right)+\frac{\partial L_{a}}{\partial \boldsymbol{X}_{a}} \delta_{a} \tag{14}
\end{align*}
$$

where $\boldsymbol{I}$ is the unit tensor and the identity $\partial \delta_{a} / \partial \boldsymbol{X}_{a}=-\partial \delta_{a} / \partial \boldsymbol{x}$ is used. For the second term in $\boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})$,

$$
\begin{align*}
& -\frac{D}{D t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}}\right)=-\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}} \frac{D \delta_{a}}{D t}-\frac{D}{D t}\left(\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right) \delta_{a} \\
& =\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}} \dot{\boldsymbol{X}}_{a} \cdot \frac{D \delta_{a}}{D \boldsymbol{x}}-\frac{D}{D t}\left(\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right) \delta_{a}=\frac{D}{D \boldsymbol{x}} \cdot\left(\dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right)-\frac{D}{D t}\left(\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right) \delta_{a} . \tag{15}
\end{align*}
$$

Thus,

$$
\begin{align*}
\boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L}) & =\frac{D}{D \boldsymbol{x}} \cdot\left(\dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{a}}{\partial \dot{\boldsymbol{X}}_{a}}-\mathcal{L}_{a} \boldsymbol{I}\right)+\left[\frac{\partial L_{a}}{\partial \boldsymbol{X}_{a}}-\frac{D}{D t}\left(\frac{\partial L_{a}}{\partial \dot{\boldsymbol{X}}_{a}}\right)\right] \delta_{a} \\
& =\frac{D}{D \boldsymbol{x}} \cdot\left(\dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{a}}{\partial \dot{\boldsymbol{X}}_{a}}-\mathcal{L}_{a} \boldsymbol{I}\right) \tag{16}
\end{align*}
$$

As expected, the integrand is not zero but a total divergence. We will refer Eq. (16) as weak Euler-Lagrange equation, which as a differential equation is equivalent to the submanifold EL equation (11). The qualifier "weak" indicates that only the spatial integral of $\boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})$ in Eq. (11) is zero [22, 23].

The weak EL equation is indispensable in establishing the connections between symmetries and local conservation laws in space-time for the classical particle-field systems under investigation. Especially, the non-vanishing right-hand-side of the weak EL equation induces a new current in the corresponding conservation laws [22, 23].

## III. SYMMETRIES AND CONSERVATION LAWS FOR PARTICLE-FIELD SYSTEMS

We now turn to the symmetries of the particle-field systems. A symmetry of the action $\mathcal{A}\left[\boldsymbol{X}_{a}, \boldsymbol{\psi}\right]$ is a group of transformation

$$
\begin{equation*}
\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \boldsymbol{\psi}\right) \longmapsto\left(\tilde{t}, \tilde{\boldsymbol{x}} ; \tilde{\boldsymbol{X}}_{a}, \tilde{\boldsymbol{\psi}}\right):=g_{\epsilon} \cdot\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \boldsymbol{\psi}\right), \tag{17}
\end{equation*}
$$

such that

$$
\begin{equation*}
\int \mathcal{L}\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \frac{d \boldsymbol{X}_{a}}{d t}, \boldsymbol{\psi}, \frac{\partial \boldsymbol{\psi}}{\partial t}, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}}\right) d t d^{3} \boldsymbol{x}=\int \mathcal{L}\left(\tilde{t}, \tilde{\boldsymbol{x}} ; \tilde{\boldsymbol{X}}_{a}, \frac{d \tilde{\boldsymbol{X}}_{a}}{d \tilde{t}}, \tilde{\boldsymbol{\psi}}, \frac{\partial \tilde{\boldsymbol{\psi}}}{\partial \tilde{t}}, \frac{\partial \tilde{\boldsymbol{\psi}}}{\partial \tilde{\boldsymbol{x}}}\right) d \tilde{t} d^{3} \tilde{\boldsymbol{x}} \tag{18}
\end{equation*}
$$

Here $g_{\epsilon}$ constitutes a continuous group of transformations parameterized by $\epsilon$ [24]. To derive the corresponding local conservation law, an infinitesimal symmetry criterion is needed. We first define the infinitesimal generator induced by the group of transformations as

$$
\begin{equation*}
\boldsymbol{v}:=\left.\frac{d}{d \epsilon}\right|_{0} g_{\epsilon} \cdot\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \boldsymbol{\psi}\right)=\xi^{t} \frac{\partial}{\partial t}+\boldsymbol{\kappa} \cdot \frac{\partial}{\partial \boldsymbol{x}}+\sum_{a} \boldsymbol{\theta}_{a} \cdot \frac{\partial}{\partial \boldsymbol{X}_{a}}+\boldsymbol{\phi} \cdot \frac{\partial}{\partial \boldsymbol{\psi}} . \tag{19}
\end{equation*}
$$

The symmetry condition (18) can be written as

$$
\begin{equation*}
\operatorname{pr}^{(1)} \boldsymbol{v}(\mathcal{L})+\mathcal{L} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi}=0 \tag{20}
\end{equation*}
$$

where $\boldsymbol{\xi}, \boldsymbol{\chi}$ are 4 D vectors in space-time, i.e., $\xi^{\mu}=\left(\xi^{t}, \boldsymbol{\kappa}\right)$ and $\chi^{\mu}=(t, \boldsymbol{x})(\mu=0,1,2,3)$ in a given coordinate system. Here, $\operatorname{pr}^{(1)} \boldsymbol{v}$, as a vector field on the jet space, is the prolongation of the vector field $\boldsymbol{v}$ on $\left\{\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \boldsymbol{\psi}\right)\right\}$,

$$
\begin{equation*}
\operatorname{pr}^{(1)} \boldsymbol{v}:=\left.\frac{d}{d \epsilon}\right|_{0}\left(\tilde{t}, \tilde{\boldsymbol{x}} ; \tilde{\boldsymbol{X}}_{a}, \frac{d \tilde{\boldsymbol{X}}_{a}}{d \tilde{t}}, \tilde{\boldsymbol{\psi}}, \frac{\partial \tilde{\boldsymbol{\psi}}}{\partial \tilde{t}}, \frac{\partial \tilde{\boldsymbol{\psi}}}{\partial \tilde{\boldsymbol{x}}}\right) \tag{21}
\end{equation*}
$$

The following expression for $\mathrm{pr}^{(1)} \boldsymbol{v}$ can be derived [24],

$$
\begin{equation*}
\operatorname{pr}^{(1)} \boldsymbol{v}=\boldsymbol{v}+\sum_{a} \boldsymbol{\theta}_{a 1} \cdot \frac{\partial}{\partial \dot{\boldsymbol{X}}_{a}}+\boldsymbol{\phi}_{\nu} \cdot \frac{\partial}{\partial\left(\frac{\partial \psi}{\partial \chi^{\nu}}\right)} \tag{22}
\end{equation*}
$$

where $\boldsymbol{\theta}_{a 1}$, and $\boldsymbol{\phi}_{\nu}$ are defined by

$$
\begin{equation*}
\boldsymbol{\theta}_{a 1}=\xi^{t} \ddot{\boldsymbol{X}}_{a}+\dot{\boldsymbol{q}}_{a}, \boldsymbol{\phi}_{\nu}=\xi^{\mu} \frac{D}{D \chi^{\nu}}\left(\frac{\partial \boldsymbol{\psi}}{\partial \chi^{\mu}}\right)+\frac{D \boldsymbol{Q}}{D \chi^{\nu}} \tag{23}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{q}_{a}=\boldsymbol{\theta}_{a}-\xi^{t} \dot{\boldsymbol{X}}_{a}, \boldsymbol{Q}=\boldsymbol{\phi}-\xi^{\mu} \frac{\partial \boldsymbol{\psi}}{\partial \chi^{\mu}} \tag{24}
\end{equation*}
$$

are the corresponding characteristics of the vector field $\boldsymbol{v}$ [24].
Sometimes the system we encounter does not admit an given symmetry, and the symmetry condition (18) is only valid for part of the Lagrangian. That is,

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}_{S}+\mathcal{F}, \tag{25}
\end{equation*}
$$

where $\mathcal{L}_{S}$ is a part of the Lagrangian density satisfying

$$
\begin{equation*}
\int \mathcal{L}_{S}\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \frac{d \boldsymbol{X}_{a}}{d t}, \boldsymbol{\psi}, \frac{\partial \boldsymbol{\psi}}{\partial t}, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}}\right) d t d^{3} \boldsymbol{x}=\int \mathcal{L}_{S}\left(\tilde{t}, \tilde{\boldsymbol{x}} ; \tilde{\boldsymbol{X}}_{a}, \frac{d \tilde{\boldsymbol{X}}}{a} \text { } \frac{\tilde{\boldsymbol{\psi}}}{d \tilde{t}}, \frac{\partial \tilde{\boldsymbol{\psi}}}{\partial \tilde{t}}, \frac{\partial \tilde{\boldsymbol{\psi}}}{\partial \tilde{\boldsymbol{x}}}\right) d \tilde{t} d^{3} \tilde{\boldsymbol{x}} \tag{26}
\end{equation*}
$$

In this situation, the infinitesimal symmetry criterion is

$$
\begin{equation*}
\operatorname{pr}^{(1)} \boldsymbol{v}(\mathcal{L})+\mathcal{L} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi}=\operatorname{pr}^{(1)} \boldsymbol{v}(\mathcal{F})+\mathcal{F} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi} \tag{27}
\end{equation*}
$$

Having derived the weak EL equation (16) and infinitesimal symmetry criterion (20) or (27), we now establish the connection between symmetries and local conservation laws. Substituting Eqs. (19), (22) and (23) into the first term of Eq. (20), we have

$$
\begin{align*}
& \operatorname{pr}^{(1)} \boldsymbol{v}(\mathcal{L})=\xi^{\mu} \frac{\partial \mathcal{L}}{\partial \chi^{\mu}}+\sum_{a} \boldsymbol{\theta}_{a} \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{X}_{a}}+\boldsymbol{\phi} \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi}} \\
& +\sum_{a}\left(\xi^{t} \ddot{\boldsymbol{X}}_{a}+\dot{\boldsymbol{q}}_{a}\right) \cdot \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}}+\left[\xi^{\mu} \frac{D}{D \chi^{\nu}}\left(\frac{\partial \boldsymbol{\psi}}{\partial \chi^{\mu}}\right)+\frac{D \boldsymbol{Q}}{D \chi^{\nu}}\right] \cdot \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial \chi^{\nu}}\right)} \\
& =\xi^{\mu} \frac{D \mathcal{L}}{D \chi^{\mu}}+\sum_{a}\left(\boldsymbol{\theta}_{a}-\xi^{t} \dot{\boldsymbol{X}}_{a}\right) \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{X}_{a}}+\sum_{a} \dot{\boldsymbol{q}}_{a} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}}+\left(\boldsymbol{\phi}-\xi^{\mu} \frac{\partial \boldsymbol{\psi}}{\partial \chi^{\mu}}\right) \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi}}+\frac{D \boldsymbol{Q}}{D \chi^{\nu}} \cdot \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial \chi^{\nu}}\right)} \\
& =\xi^{\mu} \frac{D \mathcal{L}}{D \chi^{\mu}}+\frac{D}{D t}\left(\sum_{a} \boldsymbol{q}_{a} \cdot \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}}\right)+\frac{D}{D \chi^{\nu}}\left[\boldsymbol{Q} \cdot \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial \chi^{\nu}}\right)}\right]+\sum_{a} \boldsymbol{q}_{a} \cdot \boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})+\boldsymbol{Q} \cdot \boldsymbol{E}_{\psi}(\mathcal{L}) \\
& =\frac{D}{D t}\left[\mathcal{L} \xi^{t}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)} \cdot \boldsymbol{Q}+\sum_{a} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}} \cdot \boldsymbol{q}_{a}\right]+\frac{D}{D \boldsymbol{x}} \cdot\left[\mathcal{L} \boldsymbol{\kappa}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial \boldsymbol{x}}\right)} \cdot \boldsymbol{Q}\right] \\
& +\sum_{a} \boldsymbol{q}_{a} \cdot \boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})+\boldsymbol{Q} \cdot \boldsymbol{E}_{\psi}(\mathcal{L})-\mathcal{L} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi}, \tag{28}
\end{align*}
$$

where Eq. (24) is used for the third step. Equation (20) now reads

$$
\begin{align*}
& \frac{D}{D t}\left[\mathcal{L} \xi^{t}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)} \cdot \boldsymbol{Q}+\sum_{a} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}} \cdot \boldsymbol{q}_{a}\right] \\
& +\frac{D}{D \boldsymbol{x}} \cdot\left[\mathcal{L} \boldsymbol{\kappa}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial \boldsymbol{x}}\right)} \cdot \boldsymbol{Q}\right]+\sum_{a} \boldsymbol{q}_{a} \cdot \boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})+\boldsymbol{Q} \cdot \boldsymbol{E}_{\psi}(\mathcal{L})=0 \tag{29}
\end{align*}
$$

According to the EL equation (7) for $\boldsymbol{\psi}$, the last term in Eq. (29) vanishes. However, due to the weak EL equation (16), the third term in Eq. (29) is not zero. If the characteristics $\boldsymbol{q}_{a}$ is independent of $\boldsymbol{x}$ and $\boldsymbol{\psi}$, this term can be written as a divergence form, i.e.,

$$
\begin{equation*}
\boldsymbol{q}_{a} \cdot \boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})=\frac{D}{D \boldsymbol{x}} \cdot\left[\left(\dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{a}}{\partial \dot{\boldsymbol{X}}_{a}}-\mathcal{L}_{a} \boldsymbol{I}\right) \cdot \boldsymbol{q}_{a}\right] \tag{30}
\end{equation*}
$$

which induces a new current absent in the standard field theory. Substituting Eq. (30) into Eq. (29), we finally arrive at the conservation law

$$
\begin{align*}
& \frac{D}{D t}\left[\mathcal{L} \xi^{t}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)} \cdot \boldsymbol{Q}+\sum_{a} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}} \cdot \boldsymbol{q}_{a}\right] \\
& +\frac{D}{D \boldsymbol{x}} \cdot\left[\mathcal{L} \boldsymbol{\kappa}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial \boldsymbol{x}}\right)} \cdot \boldsymbol{Q}+\sum_{a}\left(\dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{a}}{\partial \dot{\boldsymbol{X}}_{a}}-\mathcal{L}_{a} \boldsymbol{I}\right) \cdot \boldsymbol{q}_{a}\right]=0 \tag{31}
\end{align*}
$$

If the symmetry condition of the system is Eq. (27) instead, the corresponding conservation law of the system should be changed to

$$
\begin{align*}
& \frac{D}{D t}\left[\mathcal{L} \xi^{t}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial t}\right)} \cdot \boldsymbol{Q}+\sum_{a} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}} \cdot \boldsymbol{q}_{a}\right] \\
& +\frac{D}{D \boldsymbol{x}} \cdot\left[\mathcal{L} \boldsymbol{\kappa}+\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial \boldsymbol{x}}\right)} \cdot \boldsymbol{Q}+\sum_{a}\left(\dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{a}}{\partial \dot{\boldsymbol{X}}_{a}}-\mathcal{L}_{a} \boldsymbol{I}\right) \cdot \boldsymbol{q}_{a}\right]=\mathrm{pr}^{(1)} \boldsymbol{v}(\mathcal{F})+\mathcal{F} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi} \tag{32}
\end{align*}
$$

which states that the space-time divergence of the flux equals the input form the source.

## IV. SYMMETRIES AND CONSERVATION LAWS FOR KLIMONTOVICH-POISSON SYSTEM

The Klimontovich-Poisson system, as a reduced system of the Klimontovich-Maxwell system, has been applied extensively in plasma physics. The local energy-momentum conservation laws for the KP system has important implications. The action and Lagrangian density of the KP system are given by

$$
\begin{align*}
\mathcal{A} & =\int \mathcal{L}_{K P} d t d^{3} \boldsymbol{x}, \mathcal{L}_{K P}=\sum_{a} \mathcal{L}_{a}+\mathcal{L}_{F} \\
\mathcal{L}_{a} & =\left[\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}+\frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \cdot \boldsymbol{A}_{0}(\boldsymbol{x})-q_{a} \varphi\right] \delta_{a}, \mathcal{L}_{F}=\frac{(\boldsymbol{\nabla} \varphi)^{2}}{8 \pi} \tag{33}
\end{align*}
$$

where $\boldsymbol{A}_{0}$ is the vector potential for a given external magnetic field $\boldsymbol{B}_{0}=\boldsymbol{\nabla} \times \boldsymbol{A}_{0}$, and the field $\boldsymbol{\psi}$ in this case is the scalar potential $\varphi$.

As a benchmark against the result in Ref. [22], we first discuss the time translation symmetry and the energy conservation law for the KP system. Substituting Eq. (33) into Eq. (16), we immediately obtain the weak EL equation,

$$
\begin{align*}
& \boldsymbol{E}_{\boldsymbol{X}_{a}}(\mathcal{L})=\frac{\partial \mathcal{L}}{\partial \boldsymbol{X}_{a}}-\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_{a}} \\
& =\frac{D}{D \boldsymbol{x}} \cdot\left[\dot{\boldsymbol{X}}_{a}\left(m_{a} \dot{\boldsymbol{X}}_{a}+\frac{q_{a}}{c} \boldsymbol{A}_{0}(\boldsymbol{x})\right) \delta_{a}-\left(\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}+\frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \cdot \boldsymbol{A}_{0}(\boldsymbol{x})-q_{a} \varphi\right) \delta_{a} \boldsymbol{I}\right] \tag{34}
\end{align*}
$$

which is the same as the result in Ref. [22]. It is also straightforward to verify that the action of the KP system is invariant under the time translation,

$$
\begin{equation*}
\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \varphi\right) \mapsto\left(\tilde{t}, \tilde{\boldsymbol{x}} ; \tilde{\boldsymbol{X}}_{a}, \tilde{\varphi}\right)=g_{\epsilon} \cdot\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \varphi\right)=\left(t+\epsilon, \boldsymbol{x} ; \boldsymbol{X}_{a}, \varphi\right), \epsilon \in \mathbb{R} \tag{35}
\end{equation*}
$$

The infinitesimal generator of the group transformation is

$$
\begin{equation*}
\boldsymbol{v}=\frac{\partial}{\partial t} \tag{36}
\end{equation*}
$$

whose prolongation in the jet space is

$$
\begin{equation*}
\operatorname{pr}^{(1)} \boldsymbol{v}=\frac{\partial}{\partial t} \tag{37}
\end{equation*}
$$

The infinitesimal criterion (20) of the symmetry, naturally satisfied by the Lagrangian, is

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial t}=0 \tag{38}
\end{equation*}
$$

The characteristic $\boldsymbol{q}_{a}=\boldsymbol{\theta}_{a}-\xi^{t} \dot{\boldsymbol{X}}_{a}=-\dot{\boldsymbol{X}}_{a}$ is independent of $\boldsymbol{x}$ and $\varphi$. Substituting Eqs. (33) and (36) into Eq. (31), we obtain the energy conservation law,

$$
\begin{equation*}
\frac{D}{D t}\left[\frac{(\boldsymbol{\nabla} \varphi)^{2}}{8 \pi}-\sum_{a}\left(\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}+q_{a} \varphi\right) \delta_{a}\right]+\frac{D}{D \boldsymbol{x}} \cdot\left[-\frac{\boldsymbol{\nabla} \varphi}{4 \pi} \varphi_{, t}-\sum_{a}\left(\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}+q_{a} \varphi\right) \delta_{a} \dot{\boldsymbol{X}}_{a}\right]=0 \tag{39}
\end{equation*}
$$

where $\varphi_{, t} \equiv \partial_{t} \varphi$. Subtracting the identity

$$
\begin{equation*}
\frac{D}{D t}\left\{\frac{D}{D \boldsymbol{x}} \cdot\left[\varphi \frac{\partial \mathcal{L}}{\partial(\nabla \varphi)}\right]\right\}+\frac{D}{D \boldsymbol{x}} \cdot\left\{-\frac{D}{D t}\left[\varphi \frac{\partial \mathcal{L}}{\partial(\nabla \varphi)}\right]\right\}=0 \tag{40}
\end{equation*}
$$

from Eq. (391), the energy conservation is (equivalently)

$$
\begin{equation*}
\frac{D}{D t}\left[\sum_{a} \frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2} \delta_{a}+\frac{(\boldsymbol{\nabla} \varphi)^{2}}{8 \pi}\right]+\frac{D}{D \boldsymbol{x}} \cdot\left[\sum_{a}\left(\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}+q_{a} \varphi\right) \delta_{a} \dot{\boldsymbol{X}}_{a}-\frac{1}{4 \pi} \varphi \boldsymbol{\nabla} \varphi, t\right]=0 \tag{41}
\end{equation*}
$$

This agrees with the result given in Ref. [22].

We now discuss the connection between the rotational symmetry and the angular momentum conservation law of the KP system, which has not been studied previously. The Lagrangian density is first split into two parts,

$$
\begin{align*}
& \mathcal{L}_{K P}=\mathcal{L}_{S}+\mathcal{F}, \\
& \mathcal{L}_{S}=\sum_{a}\left[\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}-q_{a} \varphi\right] \delta_{a}+\frac{(\boldsymbol{\nabla} \varphi)^{2}}{8 \pi}, \mathcal{F}=\sum_{a} \frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \cdot \boldsymbol{A}_{0}(\boldsymbol{x}) \delta_{a} \tag{42}
\end{align*}
$$

where $\mathcal{L}_{S}$ is invariant under the rotational transformation and the symmetry is responsible for the conservation of local angular momentum. However, the term $\mathcal{F}$ does comply with the rotational symmetry, and it represents a torque due to the external magnetic field generating input of angular momentum to the system. We now choose a global Cartesian coordinate to describe the rotation. In this coordinate system, all vectors, such as $\boldsymbol{x}, \boldsymbol{X}_{a}$, and $\boldsymbol{A}_{0}$, are represented by $1 \times 3$ matrices. The rotational transformations of the system is defined by

$$
\begin{equation*}
\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \varphi\right) \mapsto\left(\tilde{t}, \tilde{\boldsymbol{x}} ; \tilde{\boldsymbol{X}}_{a}, \tilde{\varphi}\right)=g_{\epsilon} \cdot\left(t, \boldsymbol{x} ; \boldsymbol{X}_{a}, \varphi\right)=\left(t, \boldsymbol{R}_{\epsilon} \cdot \boldsymbol{x} ; \boldsymbol{R}_{\epsilon} \cdot \boldsymbol{X}_{a}, \varphi\right), \epsilon \in \mathbb{R} \tag{43}
\end{equation*}
$$

where $\boldsymbol{R}_{\epsilon}$ is a continuous one parameter subgroup of $\mathrm{SO}(3)$, the rotational group in the 3D Euclidean space. At $\epsilon=0, \boldsymbol{R}_{0}=\boldsymbol{I}$ is the identity matrix. Substituting Eq. (43) into Eq. (22), the infinitesimal generator and its prolongation are

$$
\begin{equation*}
\boldsymbol{v}=(\boldsymbol{\Omega} \cdot \boldsymbol{x}) \cdot \frac{\partial}{\partial \boldsymbol{x}}+\sum_{a}\left(\boldsymbol{\Omega} \cdot \boldsymbol{X}_{a}\right) \cdot \frac{\partial}{\partial \boldsymbol{X}_{a}} \tag{44}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{pr}^{(1)} \boldsymbol{v}=(\boldsymbol{\Omega} \cdot \boldsymbol{x}) \cdot \frac{\partial}{\partial \boldsymbol{x}}+\sum_{a}\left(\boldsymbol{\Omega} \cdot \boldsymbol{X}_{a}\right) \cdot \frac{\partial}{\partial \boldsymbol{X}_{a}}+\sum_{a}\left(\boldsymbol{\Omega} \cdot \dot{\boldsymbol{X}}_{a}\right) \cdot \frac{\partial}{\partial \dot{\boldsymbol{X}}_{a}}+(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \varphi) \cdot \frac{\partial}{\partial \boldsymbol{\nabla} \varphi}, \tag{45}
\end{equation*}
$$

where

$$
\begin{equation*}
\boldsymbol{\Omega}=\left.\frac{d}{d \epsilon}\right|_{0} \boldsymbol{R}_{\epsilon} \tag{46}
\end{equation*}
$$

is a $3 \times 3$ anti-symmetric matrix, i.e., an element in the Lie algebra so(3). The characteristic $\boldsymbol{q}_{a} \equiv \boldsymbol{\theta}_{a}-\xi^{t} \dot{\boldsymbol{X}}_{a}=\boldsymbol{\Omega} \cdot \boldsymbol{X}_{a}$ is independent of $\boldsymbol{x}$ and $\varphi$. Substituting Eqs. (42) and (45) into
the left-hand side of Eq. (20), we have

$$
\begin{align*}
& \operatorname{pr}^{(1)} \boldsymbol{v}\left(\mathcal{L}_{K P}\right)+\mathcal{L}_{K P} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi} \\
& =-\boldsymbol{\Omega}:\left[\boldsymbol{x} \frac{\partial \mathcal{L}_{K P}}{\partial \boldsymbol{x}}+\sum_{a} \boldsymbol{X}_{a} \frac{\partial \mathcal{L}_{K P}}{\partial \boldsymbol{X}_{a}}+\sum_{a} \dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{K P}}{\partial \dot{\boldsymbol{X}}_{a}}+\boldsymbol{\nabla} \varphi \frac{\partial \mathcal{L}_{K P}}{\partial \boldsymbol{\nabla} \varphi}\right] \\
& =-\boldsymbol{\Omega}:\left[\boldsymbol{x} \frac{\partial \mathcal{L}_{S}}{\partial \boldsymbol{x}}+\sum_{a} \boldsymbol{X}_{a} \frac{\partial \mathcal{L}_{S}}{\partial \boldsymbol{X}_{a}}+\sum_{a} \dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{S}}{\partial \dot{\boldsymbol{X}}_{a}}+\boldsymbol{\nabla} \varphi \frac{\partial \mathcal{L}_{S}}{\partial \boldsymbol{\nabla} \varphi}\right]+\mathrm{pr}^{(1)} \boldsymbol{v}(\mathcal{F})+\mathcal{F} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi} \\
& =\operatorname{pr}^{(1)} \boldsymbol{v}(\mathcal{F})+\mathcal{F} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi}-\boldsymbol{\Omega}:\left[\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a} \delta\left(\boldsymbol{x}-\boldsymbol{X}_{a}\right)+\frac{\boldsymbol{\nabla} \varphi \boldsymbol{\nabla} \varphi}{4 \pi}\right] \\
& -\sum_{a}\left(\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}-q_{a} \varphi\right) \boldsymbol{\Omega}:\left[\boldsymbol{x} \frac{\partial \delta_{a}}{\partial \boldsymbol{x}}+\boldsymbol{X}_{a} \frac{\partial \delta_{a}}{\partial \boldsymbol{X}_{a}}\right] \tag{47}
\end{align*}
$$

where operator":" between two matrices is defined to be

$$
\begin{equation*}
\boldsymbol{C}: \boldsymbol{D}=\operatorname{tr}\left(\boldsymbol{C} \cdot \boldsymbol{D}^{T}\right) \tag{48}
\end{equation*}
$$

The third term of the right-hand side of Eq. (47) is zero because $\boldsymbol{\Omega}: \boldsymbol{H}=0$ for any symmetric matrix $\boldsymbol{H}$. The last term of Eq. (47) also vanishes,

$$
\begin{equation*}
\boldsymbol{\Omega}:\left[\boldsymbol{x} \frac{\partial \delta_{a}}{\partial \boldsymbol{x}}+\boldsymbol{X}_{a} \frac{\partial \delta_{a}}{\partial \boldsymbol{X}_{a}}\right]=\left.\frac{d}{d \theta}\right|_{0} \delta\left(\boldsymbol{R}_{\theta} \cdot \boldsymbol{x}-\boldsymbol{R}_{\theta} \cdot \boldsymbol{X}_{a}\right)=\left.\frac{d}{d \theta}\right|_{0} \frac{\delta\left(\boldsymbol{x}-\boldsymbol{X}_{a}\right)}{\operatorname{det} \boldsymbol{R}_{\epsilon}}=0 . \tag{49}
\end{equation*}
$$

Equation (47) then reduces to

$$
\begin{equation*}
\operatorname{pr}^{(1)} \boldsymbol{v}\left(\mathcal{L}_{K P}\right)+\mathcal{L}_{K P} \frac{D}{D \boldsymbol{x}} \cdot \boldsymbol{\xi}=\operatorname{pr}^{(1)} \boldsymbol{v}(\mathcal{F})+\mathcal{F} \frac{D}{D \boldsymbol{x}} \cdot \boldsymbol{\xi} \tag{50}
\end{equation*}
$$

which is in the form of Eq. (27). Therefore, the corresponding conservation law assumes the form of Eq. (32). For the rotational symmetry under investigation, the right-hand side of Eq. (50) can be transformed into

$$
\begin{align*}
& \operatorname{pr}^{(1)} \boldsymbol{v}(\mathcal{F})+\mathcal{F} \frac{D}{D \boldsymbol{x}} \cdot \boldsymbol{\xi}=\operatorname{pr}^{(1)} \boldsymbol{v}(\mathcal{F})=-\boldsymbol{\Omega}:\left[\boldsymbol{x} \frac{\partial \mathcal{F}}{\partial \boldsymbol{x}}+\sum_{a} \boldsymbol{X}_{a} \frac{\partial \mathcal{F}}{\partial \boldsymbol{X}_{a}}+\sum_{a} \dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{F}}{\partial \dot{\boldsymbol{X}}_{a}}+\boldsymbol{\nabla} \varphi \frac{\partial \mathcal{F}}{\partial \boldsymbol{\nabla} \varphi}\right] \\
& =\boldsymbol{\Omega}: \sum_{a} \frac{q_{a}}{c}\left[\dot{\boldsymbol{X}}_{a} \cdot\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)^{T} \boldsymbol{x}+\boldsymbol{A}_{0}(\boldsymbol{x}) \dot{\boldsymbol{X}}_{a}\right] \delta_{a}-\sum_{a}\left[\frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \cdot \boldsymbol{A}_{0}(\boldsymbol{x})\right] \boldsymbol{\Omega}:\left[\boldsymbol{x} \frac{\partial \delta_{a}}{\partial \boldsymbol{x}}+\boldsymbol{X}_{a} \frac{\partial \delta_{a}}{\partial \boldsymbol{X}_{s p}}\right] \\
& =\boldsymbol{\Omega}: \sum_{a} \frac{q_{a}}{c}\left[\dot{\boldsymbol{X}}_{a} \cdot\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)^{T} \boldsymbol{x}+\boldsymbol{A}_{0}(\boldsymbol{x}) \dot{\boldsymbol{X}}_{a}\right] \delta_{2}, \tag{51}
\end{align*}
$$

where we used Eq. (49). The conservation law is

$$
\begin{align*}
& \frac{D}{D t}\left\{\boldsymbol{\Omega}:\left[\left(\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \delta_{2}\right) \boldsymbol{x}\right]\right\}+\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\left(\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}\right) \boldsymbol{x}+\frac{(\boldsymbol{\nabla} \varphi)^{2}}{8 \pi} \boldsymbol{I}-\frac{\boldsymbol{\nabla} \varphi \boldsymbol{\nabla} \varphi}{4 \pi}\right] \boldsymbol{x}: \boldsymbol{\Omega}\right\} \\
& +\frac{D}{D t}\left\{\boldsymbol{\Omega}:\left[\left(\sum_{a} \frac{q_{a}}{c} \boldsymbol{A}_{0}(\boldsymbol{x}) \delta_{a}\right) \boldsymbol{x}\right]\right\}+\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\left(\sum_{a} \frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \boldsymbol{A}_{0}(\boldsymbol{x}) \delta_{a}\right) \boldsymbol{x}: \boldsymbol{\Omega}\right]\right\} \\
& =\boldsymbol{\Omega}: \sum_{a} \frac{q_{a}}{c}\left[\dot{\boldsymbol{X}}_{a} \cdot\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)^{T} \boldsymbol{x}+\boldsymbol{A}_{0}(\boldsymbol{x}) \dot{\boldsymbol{X}}_{a}\right] \delta_{a} . \tag{52}
\end{align*}
$$

The last two terms on the left-hand side of Eq. (52) can be combined,

$$
\begin{align*}
& \frac{D}{D t}\left\{\boldsymbol{\Omega}:\left[\left(\sum_{a} \frac{q_{a}}{c} \boldsymbol{A}_{0}(\boldsymbol{x})\right) \boldsymbol{x}\right]\right\}+\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\left(\sum_{a} \frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \boldsymbol{A}_{0}(\boldsymbol{x}) \delta_{a}\right) \boldsymbol{x}\right]: \boldsymbol{\Omega}\right\} \\
& =\sum_{a} \frac{q_{a}}{c}\left[\dot{\boldsymbol{X}}_{a} \cdot \boldsymbol{\nabla} \boldsymbol{A}_{0}(\boldsymbol{x}) \boldsymbol{x} \delta_{a}+\boldsymbol{A}_{0}(\boldsymbol{x}) \dot{\boldsymbol{X}}_{a} \delta_{a}\right]: \boldsymbol{\Omega} \tag{53}
\end{align*}
$$

and the conservation law is simplified into

$$
\begin{align*}
& \frac{D}{D t}\left\{\boldsymbol{\Omega}:\left[\left(\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}\right) \boldsymbol{x}\right]\right\}+\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\left(\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}\right) \boldsymbol{x}+\frac{(\boldsymbol{\nabla} \varphi)^{2}}{8 \pi} \boldsymbol{I}-\frac{\boldsymbol{\nabla} \varphi \boldsymbol{\nabla} \varphi}{4 \pi}\right] \boldsymbol{x}: \boldsymbol{\Omega}\right\} \\
& =\boldsymbol{\Omega}: \sum_{a} \frac{q_{a}}{c}\left\{\dot{\boldsymbol{X}}_{a} \cdot\left[\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)^{T}-\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)\right] \boldsymbol{x}\right\} \delta_{a} \tag{54}
\end{align*}
$$

Equation (52) can be equivalently written as

$$
\begin{align*}
& \boldsymbol{\omega} \cdot\left\{\frac{D}{D t}\left[\boldsymbol{x} \times\left(\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}\right)\right]-\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}+\frac{(\boldsymbol{\nabla} \varphi)^{2}}{8 \pi} \boldsymbol{I}-\frac{\boldsymbol{\nabla} \varphi \boldsymbol{\nabla} \varphi}{4 \pi}\right] \times \boldsymbol{x}\right\}\right\} \\
& =\boldsymbol{\omega} \cdot \sum_{a} \frac{q_{s}}{c} \boldsymbol{x} \times\left\{\dot{\boldsymbol{X}}_{a} \cdot\left[\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)^{T}-\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)\right]\right\} \delta_{a} \tag{55}
\end{align*}
$$

where the vector $\boldsymbol{\omega}$ is defined as

$$
\begin{equation*}
\omega_{k} \equiv-\frac{1}{2} \sum_{i, j} \Omega_{i j} \epsilon_{i j k} \tag{56}
\end{equation*}
$$

Here, $\epsilon_{i j k}$ is the Levi-Civita symbol. Equation (56) implies

$$
\begin{equation*}
\Omega_{i j}=-\sum_{k} \omega_{k} \epsilon_{i j k} \tag{57}
\end{equation*}
$$

In Eq. (55) , the cross operator " $\times$ " is defined by

$$
\begin{equation*}
(\boldsymbol{a} \times \boldsymbol{b})_{i}=\sum_{j, k} \epsilon_{i j k} a_{j} b_{k},(\boldsymbol{C} \times \boldsymbol{a})_{i j}=\sum_{k, l} \epsilon_{i k l} C_{j k} a_{l}, i, j, k, l=1,2,3 \tag{58}
\end{equation*}
$$

for any 3 -vectors $\boldsymbol{a}, \boldsymbol{b}$ and $3 \times 3$ matrix $\boldsymbol{C}$. Due to the arbitrariness of the vector $\boldsymbol{\omega}$, Eq. (55)) implies

$$
\begin{equation*}
\frac{D}{D t}\left(\boldsymbol{x} \times \boldsymbol{g}_{K P}\right)+\frac{D}{D \boldsymbol{x}} \cdot\left(-\boldsymbol{T}_{K P} \times \boldsymbol{x}\right)=\sum_{a} \frac{q_{a}}{c} \boldsymbol{x} \times\left[\dot{\boldsymbol{X}}_{a} \times\left(\boldsymbol{\nabla} \times \boldsymbol{A}_{0}\right)\right] \delta_{2}, \tag{59}
\end{equation*}
$$

where used is made of the following identity

$$
\begin{equation*}
\dot{\boldsymbol{X}}_{a} \cdot\left[\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)^{T}-\left(\boldsymbol{\nabla} \boldsymbol{A}_{0}\right)\right]=\dot{\boldsymbol{X}}_{a} \times\left(\boldsymbol{\nabla} \times \boldsymbol{A}_{0}\right) \tag{60}
\end{equation*}
$$

The momentum density $\boldsymbol{g}_{K P}$ and stress matrix $\boldsymbol{T}_{K P}$ of the KP system in Eq. (59) are defined as

$$
\begin{equation*}
\boldsymbol{g}_{K P}=\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}, \boldsymbol{T}_{K P}=\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}+\frac{(\boldsymbol{\nabla} \varphi)^{2}}{8 \pi} \boldsymbol{I}-\frac{\boldsymbol{\nabla} \varphi \boldsymbol{\nabla} \varphi}{4 \pi} . \tag{61}
\end{equation*}
$$

## V. ROTATIONAL SYMMETRY AND ANGULAR MOMENTUM CONSERVATION LAW FOR KLIMONTOVICH-DARWIN SYSTEM

Another well-known reduced model is the Klimontovich-Darwin (KD) system [25 28]. For the KD system, the action and Lagrangian density are given by

$$
\begin{align*}
& \mathcal{A}=\int \mathcal{L}_{K D} d t d^{3} \boldsymbol{x}, \mathcal{L}_{K D}=\sum_{a} \mathcal{L}_{a}+\mathcal{L}_{F}, \\
& \mathcal{L}_{a}=\left(\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}-q_{a} \varphi+\frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \cdot \boldsymbol{A}\right) \delta_{a}, \mathcal{L}_{F}=\frac{1}{8 \pi}\left[(\boldsymbol{\nabla} \varphi)^{2}+\frac{2}{c} \boldsymbol{\nabla} \varphi \cdot \partial_{t} \boldsymbol{A}-(\boldsymbol{\nabla} \times \boldsymbol{A})^{2}\right] . \tag{62}
\end{align*}
$$

In this case, the field $\boldsymbol{\psi}=(\varphi, \boldsymbol{A})$ is the 4-potential. The vector potential $\boldsymbol{A}(t, \boldsymbol{x})$ is part of the dynamics, which is different from the external field $\boldsymbol{A}_{0}(t, \boldsymbol{x})$ of the KP system in Sec.IV. The rotational transformations of the KD system is

$$
\begin{equation*}
(\tilde{t}, \tilde{\boldsymbol{x}} ; \tilde{\boldsymbol{X}}, \tilde{\varphi}, \tilde{\mathbf{A}})=g_{\epsilon} \cdot(t, \boldsymbol{x} ; \boldsymbol{X}, \varphi, \mathbf{A})=\left(t, \boldsymbol{R}_{\epsilon} \cdot \boldsymbol{x} ; \boldsymbol{R}_{\epsilon} \cdot \boldsymbol{X}_{a}, \varphi, \boldsymbol{R}_{\epsilon} \cdot \mathbf{A}\right), \epsilon \in \mathbb{R} \tag{63}
\end{equation*}
$$

where the definition of $\boldsymbol{R}_{\epsilon}$ is same as that in Sec.IV. Note that the symmetry transformation includes the rotation of the vector potential $\boldsymbol{A}(t, \boldsymbol{x})$. The infinitesimal generator and its prolongation (63) are

$$
\begin{align*}
& \boldsymbol{v}=(\boldsymbol{\Omega} \cdot \boldsymbol{x}) \cdot \frac{\partial}{\partial \boldsymbol{x}}+\sum_{a}\left(\boldsymbol{\Omega} \cdot \boldsymbol{X}_{a}\right) \cdot \frac{\partial}{\partial \boldsymbol{X}_{a}}+(\boldsymbol{\Omega} \cdot \boldsymbol{A}) \cdot \frac{\partial}{\partial \boldsymbol{A}},  \tag{64}\\
& \operatorname{pr}^{(1)} \boldsymbol{v}=\boldsymbol{v}+\sum_{a}\left(\boldsymbol{\Omega}_{a} \cdot \dot{\boldsymbol{X}}_{a}\right) \cdot \frac{\partial}{\partial \dot{\boldsymbol{X}}_{a}}+(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \varphi) \cdot \frac{\partial}{\partial(\boldsymbol{\nabla} \varphi)}+\left(\boldsymbol{\Omega} \cdot \partial_{t} \boldsymbol{A}\right) \cdot \frac{\partial}{\partial\left(\partial_{t} \boldsymbol{A}\right)} \\
& +[\boldsymbol{\Omega} \cdot(\boldsymbol{\nabla} \boldsymbol{A})-(\boldsymbol{\nabla} \boldsymbol{A}) \cdot \boldsymbol{\Omega}]: \frac{\partial}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \tag{65}
\end{align*}
$$

The characteristic $\boldsymbol{q}_{a}=\boldsymbol{\theta}_{a}-\xi^{t} \dot{\boldsymbol{X}}_{a}=\boldsymbol{\Omega} \cdot \boldsymbol{X}_{a}$ is independent of $\boldsymbol{x}, \varphi$ and $\boldsymbol{A}$. Substituting Eqs. (62), (64) and (65) into the left-hand side of Eq. (20), we have

$$
\begin{align*}
& \operatorname{pr}^{(1)} \boldsymbol{v}\left(\mathcal{L}_{K D}\right)+\mathcal{L}_{K D} \frac{D}{D \boldsymbol{\chi}} \cdot \boldsymbol{\xi} \\
& =-\boldsymbol{\Omega}:\left\{\boldsymbol{x} \frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{x}}+\sum_{a} \boldsymbol{X}_{a} \frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{X}_{a}}+\boldsymbol{A} \frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}}+\sum_{a} \dot{\boldsymbol{X}}_{a} \frac{\partial \mathcal{L}_{K D}}{\partial \dot{\boldsymbol{X}}_{s p}}+\boldsymbol{\nabla} \varphi \frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \varphi)}\right. \\
& =-\boldsymbol{\Omega}:\left\{\sum_{a}\left(\frac{1}{2} m_{a} \dot{\boldsymbol{X}}_{a}^{2}-q_{a} \varphi+\frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \cdot \boldsymbol{A}\right)\left(\boldsymbol{x} \frac{\partial \delta_{a}}{\partial \boldsymbol{x}}+\boldsymbol{X}_{a} \frac{\partial \delta_{a}}{\partial \boldsymbol{X}_{a}}\right)\right. \\
& +\sum_{a}\left[\frac{q_{a}}{c}\left(\boldsymbol{A} \dot{\boldsymbol{X}}_{a}+\dot{\boldsymbol{X}}_{a} \boldsymbol{A}\right) \delta_{a}+m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}\right]+\frac{1}{4 \pi}\left[\boldsymbol{\nabla} \varphi \boldsymbol{\nabla} \varphi+\frac{1}{c}\left(\boldsymbol{\nabla} \varphi \partial_{t} \boldsymbol{A}+\partial_{t} \boldsymbol{A} \boldsymbol{\nabla} \varphi\right)\right] \\
& \left.+\frac{1}{4 \pi}\left[(\boldsymbol{\nabla} \boldsymbol{A})-(\boldsymbol{\nabla} \boldsymbol{A})^{T}\right] \cdot\left[(\boldsymbol{\nabla} \boldsymbol{A})-(\boldsymbol{\nabla} \boldsymbol{A})^{T}\right]\right\}, \tag{66}
\end{align*}
$$

where used is made of the following equations

$$
\begin{equation*}
\frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})}=-\frac{1}{4 \pi}(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A}) \cdot \boldsymbol{\epsilon}=-\frac{1}{4 \pi} \boldsymbol{\epsilon} \cdot(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A}), \boldsymbol{\epsilon} \cdot(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A})=(\boldsymbol{\nabla} \boldsymbol{A})-(\boldsymbol{\nabla} \boldsymbol{A})^{T} . \tag{67}
\end{equation*}
$$

The first term on the right-hand side of Eq. (66) is zero because of Eq. (49). The last three terms on the right-hand side of Eq. (66) also vanish because they are the traces of matrix products between a symmetric and an anti-symmetric matrices. The vanishing right-handside of Eq. (66) verifies that Eq. (63) is a symmetry of the system. Substituting $\boldsymbol{\xi}, \boldsymbol{\theta}_{a}$ and $\boldsymbol{\phi}$ in Eq. (64) into Eq. (31), we obtain the angular momentum conservation law of the rotational symmetry for the KD system,

$$
\begin{align*}
& \frac{D}{D t}\left\{-\boldsymbol{\Omega}: \boldsymbol{x}\left[\sum_{a}\left(m_{a} \dot{\boldsymbol{X}}_{a}+\frac{q_{a}}{c} \boldsymbol{A}\right) \delta_{a}-\frac{1}{4 \pi c} \boldsymbol{\nabla} \boldsymbol{A} \cdot \boldsymbol{\nabla} \varphi\right]-\frac{1}{4 \pi c} \boldsymbol{\Omega}:(\boldsymbol{A} \boldsymbol{\nabla} \varphi)\right\} \\
& +\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\frac{1}{8 \pi}\left[(\boldsymbol{\nabla} \varphi)^{2}+\frac{2}{c} \boldsymbol{\nabla} \varphi \cdot \partial_{t} \boldsymbol{A}-(\boldsymbol{\nabla} \times \boldsymbol{A})^{2}\right] \boldsymbol{I}+\sum_{a}\left(m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a}+\frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \boldsymbol{A}\right) \delta_{a}\right.\right. \\
& \left.\left.-\frac{1}{4 \pi} \boldsymbol{\epsilon}:\left[(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A})(\boldsymbol{\nabla} \boldsymbol{A})^{T}\right]-\frac{1}{4 \pi}\left(\boldsymbol{\nabla} \varphi+\frac{1}{c} \partial_{t} \boldsymbol{A}\right) \boldsymbol{\nabla} \varphi\right] \boldsymbol{x}: \boldsymbol{\Omega}-\frac{1}{4 \pi}[\boldsymbol{\epsilon} \cdot(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A}) \boldsymbol{A}]: \boldsymbol{\Omega}\right\}=0 . \tag{68}
\end{align*}
$$

To put the conservation law into a symmetric form, we add the following identity

$$
\begin{equation*}
\frac{D}{D t}\left\{\frac{D}{D \boldsymbol{x}} \cdot\left[\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}} \boldsymbol{A} \boldsymbol{x}\right]\right\}: \boldsymbol{\Omega}+\frac{D}{D \boldsymbol{x}} \cdot\left\{-\frac{D}{D t}\left[\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}} \boldsymbol{A} \boldsymbol{x}\right]\right\}: \boldsymbol{\Omega}=0 \tag{69}
\end{equation*}
$$

to Eq. (68) to get

$$
\begin{align*}
& \frac{D}{D t}\left\{-\boldsymbol{\Omega}: \boldsymbol{x}\left[\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}+\frac{(-\boldsymbol{\nabla} \varphi) \times(\boldsymbol{\nabla} \times \boldsymbol{A})}{4 \pi c}-\frac{1}{4 \pi c} \boldsymbol{A} \frac{D}{D \boldsymbol{x}} \cdot\left(\frac{1}{c} \partial_{t} \boldsymbol{A}\right)\right]\right\} \\
& +\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\frac{1}{8 \pi}\left[(\boldsymbol{\nabla} \varphi)^{2}+\frac{2}{c} \boldsymbol{\nabla} \varphi \cdot \boldsymbol{A}_{, t}+(\boldsymbol{\nabla} \times \boldsymbol{A})\right] \boldsymbol{I}-\frac{1}{4 \pi}\left(-\boldsymbol{\nabla} \varphi-\frac{1}{c} \boldsymbol{A}_{, t}\right)\left(-\boldsymbol{\nabla} \varphi-\frac{1}{c} \boldsymbol{A}_{, t}\right)\right.\right. \\
& \left.\left.-\frac{1}{4 \pi}(\boldsymbol{\nabla} \times \boldsymbol{A})(\boldsymbol{\nabla} \times \boldsymbol{A})+\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}+\frac{1}{4 \pi c^{2}} \boldsymbol{A}_{, t} \boldsymbol{A}_{, t}\right] \boldsymbol{x}: \boldsymbol{\Omega}\right\} \tag{70}
\end{align*}
$$

The detailed calculation of this symmetrization process is given in Appendix A.
Using the relations between $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ and the arbitrariness of $\boldsymbol{\omega}$, the angular momentum conservation law (70) of the KD system can be equivalently rewritten as

$$
\begin{equation*}
\frac{D}{D t}\left(\boldsymbol{x} \times \boldsymbol{g}_{K D}\right)+\frac{D}{D \boldsymbol{x}} \cdot\left(-\boldsymbol{T}_{K D} \times \boldsymbol{x}\right)=0 \tag{71}
\end{equation*}
$$

where the momentum density $\boldsymbol{g}_{K D}$ and the stress matrix $\boldsymbol{T}_{K D}$ are defined by

$$
\begin{align*}
& \boldsymbol{g}_{K D}=\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}+\frac{(-\boldsymbol{\nabla} \varphi) \times \boldsymbol{B}}{4 \pi c}-\frac{1}{4 \pi c^{2}} \boldsymbol{A} \partial_{t}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) \\
& \boldsymbol{T}_{K D}=\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}+\frac{(\boldsymbol{\nabla} \varphi)^{2}+2 \boldsymbol{\nabla} \varphi \cdot \boldsymbol{A}_{, t} / c+\boldsymbol{B}^{2}}{8 \pi} \boldsymbol{I}-\frac{\boldsymbol{E} \boldsymbol{E}+\boldsymbol{B} \boldsymbol{B}-\boldsymbol{A}_{, t} \boldsymbol{A}_{, t} / c^{2}}{4 \pi} \tag{72}
\end{align*}
$$

In obtaining Eq. (72), the following identities were used,

$$
\begin{align*}
& \left(\boldsymbol{\nabla} \boldsymbol{A}-(\boldsymbol{\nabla} \boldsymbol{A})^{T}\right) \cdot\left(-\boldsymbol{\nabla} \varphi-\frac{1}{c} \partial_{t} \boldsymbol{A}\right)=\left(-\boldsymbol{\nabla} \varphi-\frac{1}{c} \partial_{t} \boldsymbol{A}\right) \times(\boldsymbol{\nabla} \times \boldsymbol{A})  \tag{73}\\
& \boldsymbol{\epsilon}:\left[(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A})\left((\boldsymbol{\nabla} \boldsymbol{A})-(\boldsymbol{\nabla} \boldsymbol{A})^{T}\right)\right]=(\boldsymbol{\nabla} \times \boldsymbol{A})^{2} \boldsymbol{I}-(\boldsymbol{\nabla} \times \boldsymbol{A})(\boldsymbol{\nabla} \times \boldsymbol{A}) .
\end{align*}
$$

If we choose the Coulomb gauge, i.e., $\boldsymbol{\nabla} \cdot \boldsymbol{A}=0$, the momentum density is

$$
\begin{equation*}
\boldsymbol{g}_{K D}=\sum_{a} m_{a} \dot{\boldsymbol{X}}_{a} \delta_{a}+\frac{-\boldsymbol{\nabla} \varphi \times \boldsymbol{B}}{4 \pi c} \tag{74}
\end{equation*}
$$

which is same as the result given in Ref. [25].

## VI. CONCLUSIONS

In this study, we developed a general field theory for classical particle-field systems, and established the connections between general symmetries and local conservation laws in spacetime for the systems. Compared with the standard classical field theory, the distinguish feature of the classical particle-field systems is that the particles and fields reside on different manifolds. The fields are defined on the 4D space-time, whereas each particle's trajectory is defined on the 1D time-axis. As a consequence, the standard Noether's procedure for deriving local conservation laws from symmetries do not apply straightforwardly without modification. To overcome this difficulty, a weak Euler-Lagrange equation for particles is developed on the 4 D space-time, which plays a pivotal role in establishing the connections between symmetries and local conservation laws in space-time. Especially, the non-vanishing Euler derivative in the weak EL equation generates a new current in the corresponding conservation laws.

Several examples from plasma physics are studied as special cases of the general field theory. As a benchmark, the time translation symmetry of the Klimontovich-Poisson (KP) system and the corresponding local energy conservation law were obtained by the general theory and compared with the results in Ref. [22]. As new applications, the relations between the rotational symmetry and angular momentum conservation for the KP system and Klimontovich-Darwin (KD) system are established. For the KP system, the conservation law is manifested as the balance between space-time divergence of the angular momentum flux and the input due to the torque of the external magnetic field. For the KD system, it is found that the rotational symmetry admitted by the system needs to include the rotation
of the vector potential. Such a rotation is a representation of the rotational group in the fiber of the vector bundle at each space-time location.

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## Appendix A: the symmetrization process of Eq. (68)

In this appendix, we give a detailed derivation of Eq. (70). The first term of Eq. (69) is

$$
\begin{align*}
& \frac{D}{D t}\left\{\frac{D}{D \boldsymbol{x}} \cdot\left[\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}} \boldsymbol{A} \boldsymbol{x}\right]\right\}: \boldsymbol{\Omega}=\boldsymbol{\Omega}: \frac{D}{D t}\left\{\frac{D}{D \boldsymbol{x}} \cdot\left[\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}}\right] \boldsymbol{A} \boldsymbol{x}+\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}} \cdot \boldsymbol{\nabla} \boldsymbol{A} \boldsymbol{x}+\boldsymbol{A} \frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}}\right\} \\
& =\boldsymbol{\Omega}: \frac{D}{D t}\left\{\frac{1}{c} \frac{D}{D \boldsymbol{x}} \cdot\left(\frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \varphi)}-\frac{1}{4 \pi c} \partial_{t} \boldsymbol{A}\right) \boldsymbol{A} \boldsymbol{x}+\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A} \boldsymbol{A}_{, t}} \cdot \boldsymbol{\nabla} \boldsymbol{A} \boldsymbol{x}+\boldsymbol{A} \frac{\partial \mathcal{L}_{K D}}{\partial\left(\boldsymbol{A}_{, t}\right)}\right\} \\
& =\boldsymbol{\Omega}: \frac{D}{D t}\left\{\frac{1}{c} \frac{\partial \mathcal{L}_{K D}}{\partial \varphi} \boldsymbol{A} \boldsymbol{x}-\frac{1}{4 \pi c} \frac{D}{D \boldsymbol{x}} \cdot\left(\frac{1}{c} \partial_{t} \boldsymbol{A}\right) \boldsymbol{A} \boldsymbol{x}+\frac{1}{4 \pi c} \boldsymbol{\nabla} \varphi \cdot \boldsymbol{\nabla} \boldsymbol{A} \boldsymbol{x}+\frac{1}{4 \pi c} \boldsymbol{A} \boldsymbol{\nabla} \varphi\right\} \\
& =\frac{D}{D t}\left\{-\boldsymbol{\Omega}: \boldsymbol{x}\left[-\sum_{a} \frac{q_{a}}{c} \boldsymbol{A} \delta_{a}-\frac{1}{4 \pi c} \boldsymbol{A} \frac{D}{D \boldsymbol{x}} \cdot\left(\frac{1}{c} \partial_{t} \boldsymbol{A}\right)+\frac{1}{4 \pi c}(\boldsymbol{\nabla} \boldsymbol{A})^{T} \cdot \boldsymbol{\nabla} \varphi\right]+\frac{1}{4 \pi c} \boldsymbol{\Omega}: \boldsymbol{A} \boldsymbol{\nabla} \varphi\right\} . \tag{A1}
\end{align*}
$$

The second term of Eq. (69) can be written as

$$
\begin{align*}
& \frac{D}{D \boldsymbol{x}} \cdot\left\{-\frac{D}{D t}\left[\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{t,}} \boldsymbol{A} \boldsymbol{x}\right]\right\}: \boldsymbol{\Omega} \\
& =-\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}} \boldsymbol{A}+\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}}\left(\partial_{t} \boldsymbol{A}\right)-\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \boldsymbol{A}\right] \boldsymbol{x}\right\}: \boldsymbol{\Omega} \\
& =-\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}} \boldsymbol{A}+\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}}\left(\partial_{t} \boldsymbol{A}\right)\right] \boldsymbol{x}-\frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \cdot \boldsymbol{\nabla}(\boldsymbol{A} \boldsymbol{x})\right. \\
& \left.-\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \boldsymbol{A} \boldsymbol{x}+\frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \cdot \boldsymbol{\nabla}(\boldsymbol{A} \boldsymbol{x})\right\}: \boldsymbol{\Omega} \\
& =-\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}} \boldsymbol{A}+\frac{\partial \mathcal{L}_{K D}}{\partial \boldsymbol{A}_{, t}}\left(\partial_{t} \boldsymbol{A}\right)-\frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \cdot(\boldsymbol{\nabla} \boldsymbol{A})\right] \boldsymbol{x}+\frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \boldsymbol{A}\right. \\
& \left.-\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})}(\boldsymbol{A} \boldsymbol{x})+\frac{\partial \mathcal{L}_{K D}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \cdot \boldsymbol{\nabla}(\boldsymbol{A} \boldsymbol{x})\right\}: \boldsymbol{\Omega} . \tag{A2}
\end{align*}
$$

The EL equations for $\varphi$ and $\boldsymbol{A}$ have been used in the above derivation. Substituting the Lagrangian density $\mathcal{L}_{K D}$ and $\mathcal{L}_{a}$ in Eq. (62) into Eq. ((A2)), we have

$$
\begin{align*}
& \frac{D}{D \boldsymbol{x}} \cdot\left\{-\frac{D}{D t}\left[\frac{\partial \mathcal{L}_{K M}}{\partial \boldsymbol{A}_{, t}} \boldsymbol{A} \boldsymbol{x}\right]\right\}: \boldsymbol{\Omega} \\
& =\frac{D}{D \boldsymbol{x}} \cdot\left\{\left[-\sum_{a} \frac{q_{a}}{c} \dot{\boldsymbol{X}}_{a} \boldsymbol{A} \delta_{a}-\frac{1}{4 \pi c} \boldsymbol{\nabla} \varphi\left(\partial_{t} \boldsymbol{A}\right)+\frac{1}{4 \pi} \boldsymbol{\epsilon}:[(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A})(\boldsymbol{\nabla} \boldsymbol{A})]\right] \boldsymbol{x}: \boldsymbol{\Omega}\right. \\
& \left.+\frac{1}{4 \pi} \boldsymbol{\epsilon} \cdot(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A}) \boldsymbol{A}: \boldsymbol{\Omega}\right\} \tag{A3}
\end{align*}
$$

The last two terms in Eq. (A3) vanish, i.e.,

$$
\begin{align*}
& -\frac{D}{D \boldsymbol{x}} \cdot\left\{-\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}_{K M}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})}(\boldsymbol{A} \boldsymbol{x})+\frac{\partial \mathcal{L}_{K M}}{\partial(\boldsymbol{\nabla} \boldsymbol{A})} \cdot \boldsymbol{\nabla}(\boldsymbol{A} \boldsymbol{x})\right\} \\
& =\frac{1}{4 \pi} \boldsymbol{\nabla} \cdot\{-\boldsymbol{\nabla} \cdot[\boldsymbol{\epsilon} \cdot(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A})](\boldsymbol{A} \boldsymbol{x})+[\boldsymbol{\epsilon} \cdot(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A})] \cdot \boldsymbol{\nabla}(\boldsymbol{A} \boldsymbol{x})\} \\
& =\frac{1}{4 \pi} \boldsymbol{\nabla} \cdot\{\boldsymbol{\epsilon}:[\boldsymbol{\nabla}(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A})](\boldsymbol{A} \boldsymbol{x})-\boldsymbol{\epsilon}:[(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A}) \cdot \boldsymbol{\nabla}(\boldsymbol{A} \boldsymbol{x})]\} \\
& =\frac{1}{4 \pi} \boldsymbol{\nabla} \cdot\{\boldsymbol{\nabla} \times[(\boldsymbol{\epsilon}: \boldsymbol{\nabla} \boldsymbol{A})] \boldsymbol{A} \boldsymbol{x}\}=0 . \tag{A4}
\end{align*}
$$

Substituting Eqs. (A1) and (A3) into Eq. (69), and adding it to Eq. (68), we obtain Eq. (70).
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