Resistive Tearing Layer Matching Conditions, Revisited

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Resistive Instabilities as Matched Asymptotic Expansions

- A. H. Glasser, Z. R. Wang, and J.-K. Park, Phys. Plasmas **23**, 112506 (2016).
- At each singular surface $q = m/n$, the small-x asymptotic solutions of the ideal outer region are matched to the large-x asymptotic solutions of the resistive inner region.
- The determinant of the global matching matrix is a dispersion relation for the global complex growth rate and eigenfunctions.
- Verification against the straight-through MARS-F code reveals regimes of excellent agreement and other regimes of strong disagreement.
- Careful study reveals that the asymptotic expansion used to determine the inner region matching data fails to converge asymptotically in the cases of strong disagreement.
- A new expansion procedure is found, applying the results of W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, 1965.

Resistive Inner Region Equations

A.H. Glasser, J.M. Greene, and J.L. Johnson, Phys. Fluids 18, 7, 875 (1975).

Scaled Equations

$$\Psi_{xx} - H\Upsilon_x - Q(\Psi - x\Xi) = 0 \qquad (1)$$

$$Q^2\Xi_{xx} - Qx^2\Xi + Qx\Psi + (E+F)\Upsilon + H\Psi_x = 0 \qquad (2)$$

$$Q\Upsilon_{xx} - x^2\Upsilon + x\Psi + Q^2\left[G\left(\Xi - \Upsilon\right) - K\left(E\Xi + F\Upsilon + H\Psi_x\right)\right] = 0 \qquad (3)$$

Normalized Variables

$\Psi =$ perturbed parallel vector potential

$\Xi =$ perturbed electrostatic potential

$\Upsilon =$ perturbed fluid pressure

$X =$ distance from singular surface

$Q =$ complex growth rate

$E, F, G, H, K =$ equilibrium parameters

Physical Interpretation

(1) Ohm's Law

(2) Quasineutrality

(3) AdiabaticPressureLaw

Coupled System of 2nd-Order Equations

A. H. Glasser, Z. R. Wang, and J.-K. Park, Phys. Plasmas 23, 112506 (2016).

Matrix Form

$$\mathbf{\Psi} \equiv \begin{pmatrix} \Psi \\ \Xi \\ \Upsilon \end{pmatrix}, \quad \mathsf{A}\mathbf{\Psi}'' + \mathsf{B}\mathbf{\Psi}' + \mathsf{C}\mathbf{\Psi} = \mathbf{0}$$

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q^2 & 0 \\ 0 & 0 & Q \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 0 & 0 & -H \\ H & 0 & 0 \\ -KHQ^2 & 0 & 0 \end{pmatrix}$$

$$\mathsf{C} = \begin{pmatrix} -Q & xQ & 0 \\ Qx & -Qx^2 & E+F \\ x & (G-KE)Q^2 & -x^2-(G+KF)Q^2 \end{pmatrix}$$

Shearing Transformation

$$\mathbf{\Psi} = \mathsf{R}\mathbf{u}, \quad \mathbf{\Psi}' = \mathsf{R}\mathbf{u}' + \mathsf{R}'\mathbf{u}, \quad \mathbf{\Psi}'' = \mathsf{R}\mathbf{u}'' + 2\mathsf{R}'\mathbf{u}' + \mathsf{R}''\mathbf{u}$$

$$\mathsf{R} = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{R}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{R}'' = \mathbf{0}$$

$$\mathsf{S} = \begin{pmatrix} 1/x & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1/x^2 \end{pmatrix}$$

Final Form of Matrix Equations

Transformed Equations

$$\mathsf{A}\Psi'' + \mathsf{B}\Psi' + \mathsf{C}\Psi = 0$$

$$\mathsf{S}\left[\mathsf{A}(\mathbf{Ru})'' + \mathsf{B}(\mathbf{Ru})' + \mathsf{C}(\mathbf{Ru})\right] = 0$$

$$\mathsf{S}\left[(\mathsf{AR})\mathbf{u}'' + (2\mathsf{AR}' + \mathsf{BR})\mathbf{u}' + (\mathsf{AR}'' + \mathsf{BR}' + \mathsf{CR})\mathbf{u}\right] = 0$$

$$\bar{\mathsf{A}}\mathbf{u}'' + \bar{\mathsf{B}}\mathbf{u}' + \bar{\mathsf{C}}\mathbf{u} = 0$$

Transformed Matrices

$$\bar{\mathsf{A}} = \mathsf{SAR} = x^2\left(\bar{\mathsf{A}}_0 + \bar{\mathsf{A}}_1 x^{-2} + \bar{\mathsf{A}}_2 x^{-4}\right) = \begin{pmatrix} 1 & 0 & 0 \\ x^2 & Q^2 & 0 \\ 0 & 0 & Q/x^2 \end{pmatrix}$$

$$\bar{\mathsf{B}} = \mathsf{S}(2\mathsf{AR}' + \mathsf{BR}) = x\left(\bar{\mathsf{B}}_0 + \bar{\mathsf{B}}_1 x^{-2}\right) = \begin{pmatrix} 2/x & 0 & -H/x \\ (2+H)x & 0 & -Hx \\ -HKQ^2/x & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \bar{\mathsf{C}} &= \mathsf{S}(\mathsf{AR}'' + \mathsf{BR}' + \mathsf{CR}) = \bar{\mathsf{C}}_0 + \bar{\mathsf{C}}_1 x^{-2} \\ &= \begin{pmatrix} -Q & Q & 0 \\ H & 0 & E+F \\ 1 - KHQ^2/x^2 & (G-KE)Q^2/x^2 & -1-(G+KF)Q^2/x^2 \end{pmatrix} \end{aligned}$$

Power Series Expansion

Descending Power Series Solutions, $x \to \infty$

$$
\begin{aligned}
\mathbf{u} &= x^{\mu} \sum_{j=0}^{\infty} x^{-2j} \mathbf{u}_j \\
\mathbf{u}' &= x^{\mu-1} \sum_{j=0}^{\infty} x^{-2j} (\mu - 2j) \mathbf{u}_j \\
\mathbf{u}'' &= x^{\mu-2} \sum_{j=0}^{\infty} x^{-2j} (\mu - 2j)(\mu - 2j - 1) \mathbf{u}_j
\end{aligned}
$$

Power Series Equations

$$
\begin{aligned}
x^{-\mu} \mathbf{L}\mathbf{u} &= x^{-\mu} \left(\bar{\mathbf{A}}\mathbf{u}'' + \bar{\mathbf{B}}\mathbf{u}' + \bar{\mathbf{C}}\mathbf{u} \right) \\
&= \sum_{j=0}^{\infty} x^{-2j} \left\{ \left[(\mu - 2j)(\mu - 2j - 1)\bar{\mathbf{A}}_0 + (\mu - 2j)\bar{\mathbf{B}}_0 + \bar{\mathbf{C}}_0 \right] \mathbf{u}_j \right. \\
&\quad + \left[(\mu - 2j + 2)(\mu - 2j + 1)\bar{\mathbf{A}}_1 + (\mu - 2j + 2)\bar{\mathbf{B}}_1 + \bar{\mathbf{C}}_1 \right] \mathbf{u}_{j-1} \\
&\quad \left. + (\mu - 2j + 4)(\mu - 2j + 3)\bar{\mathbf{A}}_2 \mathbf{u}_{j-2} \right\} = 0
\end{aligned}
$$

$$
\begin{aligned}
&\left[(\mu - 2j)(\mu - 2j - 1)\bar{\mathbf{A}}_0 + (\mu - 2j)\bar{\mathbf{B}}_0 + \bar{\mathbf{C}}_0 \right] \mathbf{u}_j \\
&+ \left[(\mu - 2j + 2)(\mu - 2j + 1)\bar{\mathbf{A}}_1 + (\mu - 2j + 2)\bar{\mathbf{B}}_1 + \bar{\mathbf{C}}_1 \right] \mathbf{u}_{j-1} \\
&+ (\mu - 2j + 4)(\mu - 2j + 3)\bar{\mathbf{A}}_2 \mathbf{u}_{j-2} = 0
\end{aligned}
$$

Order-by-Order Solution

Zeroth Order Equations

$$\bar{\mathbf{L}}_0 \equiv \mu(\mu-1)\bar{\mathbf{A}}_0 + \mu\bar{\mathbf{B}}_0 + \bar{\mathbf{C}}_0, \quad \bar{\mathbf{L}}_0\mathbf{u}_0 = 0, \quad \det\bar{\mathbf{L}}_0 = 0$$

$$\bar{\mathbf{A}}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \bar{\mathbf{B}}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 2+H & 0 & -H \\ 0 & 0 & 0 \end{pmatrix}, \ \bar{\mathbf{C}}_0 = \begin{pmatrix} -Q & Q & 0 \\ H & 0 & E+F \\ 1 & 0 & -1 \end{pmatrix}$$

$$\bar{\mathbf{L}}_0 = \begin{pmatrix} -Q & Q & 0 \\ \mu(\mu-1)+(2+H)\mu+H & 0 & E+F-\mu H \\ 1 & 0 & -1 \end{pmatrix}$$

$$\det\bar{\mathbf{L}}_0 = Q\left(\mu^2+\mu+E+F+H\right) = 0$$

$$\mu = -\frac{1}{2} \pm \sqrt{-D_I}, \quad D_I = E+F+H-\frac{1}{4}, \quad \mathbf{u}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

First Order Equations

$$\begin{aligned}
&\left[(\mu-2)(\mu-3)\bar{\mathbf{A}}_0 + (\mu-2)\bar{\mathbf{B}}_0 + \bar{\mathbf{C}}_0\right]\mathbf{u}_1 \\
&\qquad = -\left[\mu(\mu-1)\bar{\mathbf{A}}_1 + \mu\bar{\mathbf{B}}_1 + \bar{\mathbf{C}}_1\right]\mathbf{u}_0
\end{aligned}$$

Higher-Order Equations

$$\begin{aligned}
&\left[(\mu-2j)(\mu-2j-1)\bar{\mathbf{A}}_0 + (\mu-2j)\bar{\mathbf{B}}_0 + \bar{\mathbf{C}}_0\right]\mathbf{u}_j \\
&\quad = -\left[(\mu-2j+2)(\mu-2j+1)\bar{\mathbf{A}}_1 + (\mu-2j+2)\bar{\mathbf{B}}_1 + \bar{\mathbf{C}}_1\right]\mathbf{u}_{j-1} \\
&\qquad - (\mu-2j+4)(\mu-2j+3)\bar{\mathbf{A}}_2\mathbf{u}_{j-2}
\end{aligned}$$

Problems with the 2^{nd}-Order Expansion

- For the simplest cases tested, the power series solutions converge well and give results in good agreement with the MARS-F code.
- In other regimes, the inner region solution fails to overlap with the outer region solutions.
- In these regimes, the power series solution fails to provide asymptotically convergent solutions.
- In comparison with textbook treatments of this problem, the coupled 2^{nd}-order equations used here are nonstandard. Textbook treatments use a coupled 1^{st}-order formulation.
- A new expansion procedure is found, applying the results of W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, 1965.
- This procedure is based on a formulation in terms of a coupled set of 1^{st}-order equations in matrix form.
- The equations are found to be degenerate, leading to a difficult solution procedure, treated only by Wasow, outlined here.

Comparison of Growth Rate Between Resistive DCON and MARS-F

Aspect ratio case A=5
B_0=1T, R_0=2m, $\eta=9\times10^{-10}$
q_0=1.1, q_a=2.2, β=0.0035, β_N=0.73

Δ'(DCON) = 1360 > 0

γ(DCON) = 11.74 s^{-1}

γ(MARS) = 11.99 s^{-1}

Lower aspect ratio

Aspect ratio case **A=2**
B_0=1T, R_0=2m, $\eta=7\times10^{-9}$
q_0=1.1, q_a=2.43, β=0.009, β_N=0.7

Δ'(DCON) = 17.87 > 0

γ(DCON) = - 25.98 - 29.7i s^{-1} (stable)

γ(MARS) = 124 s^{-1} (unstable)

Increase pressure

q_0=1.1, q_a=2.29 **β=0.0066, β_N=1.43**

Δ'(DCON) = 225 > 0

γ(DCON) = - 42.17 + 71.6 s^{-1}(stable)

γ(MARS) = 5.11 s^{-1} (unstable)

Simulated equilibrium with circular cross session is used for benchmark.

GGJ shows a much stronger stabilizing effect than MARS-F while increase pressure and reduce aspect ratio.

Coupled 1st-Order System

Matrix Form

$$\boldsymbol{\Psi} \equiv \begin{pmatrix} x\Psi \\ \Xi \\ \Upsilon \end{pmatrix}, \quad \mathbf{u} \equiv \begin{pmatrix} \boldsymbol{\Psi} \\ \boldsymbol{\Psi}'/x \end{pmatrix}, \quad \mathbf{u}' = x\mathbf{A}\mathbf{u}$$

Coefficient Matrix

$$\mathbf{A} = \mathbf{A}_0 + x^{-2}\mathbf{A}_1 + x^{-4}\mathbf{A}_2$$

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -Q & 0 & 0 & 0 & H \\ 0 & 1/Q & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/Q & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ H/Q^2 & 0 & 0 & 0 & 0 & 0 \\ -HKQ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ Q & 0 & 0 & 1 & 0 & 0 \\ -1/Q & 0 & -(E+F)/Q^2 & -H/Q^2 & -1 & 0 \\ -1/Q & -(G-KE)Q & (G+KF)Q & HKQ & 0 & -1 \end{pmatrix}$$

Transformation to Jordan Canonical Form

with help from Mathematica

$$\mathbf{u} = \mathbf{T}\mathbf{v}, \quad \mathbf{v}' = x\mathbf{J}\mathbf{v}, \quad \mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \quad \lambda \equiv Q^{-1/2}$$

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & HQ & Q^{5/2} & HQ & -Q^{5/2} \\ 0 & 0 & 0 & -Q^{1/2} & 0 & Q^{1/2} \\ 0 & 0 & -Q^{1/2} & 0 & Q^{1/2} & 0 \\ 0 & 1 & -HQ^{1/2} & -Q^2 & HQ^{1/2} & -Q^2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & Q^2 & 0 & 0 & 0 & -HQ \\ 0 & 0 & -H & 1 & Q^2 & 0 \\ 0 & 0 & -1/2Q^{1/2} & 0 & 0 & 1/2 \\ 0 & -1/2Q^{1/2} & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2Q^{1/2} & 0 & 0 & 1/2 \\ 0 & 1/2Q^{1/2} & 0 & 0 & 1/2 & 0 \end{pmatrix}$$

$$\mathbf{J}_0 = \mathbf{T}^{-1}\mathbf{A}_0\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$\mathbf{J}_1 = \mathbf{T}^{-1}\mathbf{A}_1\mathbf{T}, \quad \mathbf{J}_2 = \mathbf{T}^{-1}\mathbf{A}_2\mathbf{T}$$

Schur Complement

Separates Power-Like from Exponential Solutions

Partition of Linear Equations

$$\mathrm{x} = \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix}, \quad \mathrm{b} = \begin{pmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathrm{A}_{12} \\ \mathbf{A}_{21} & \mathrm{A}_{22} \end{pmatrix}$$

$$\mathbf{A}\mathrm{x} = \mathrm{b}$$

$$\begin{aligned} \mathbf{A}_{11}\mathrm{x}_1 + \mathbf{A}_{12}\mathrm{x}_2 &= \mathrm{b}_1 \\ \mathbf{A}_{21}\mathrm{x}_1 + \mathbf{A}_{22}\mathrm{x}_2 &= \mathrm{b}_2 \end{aligned}$$

Elimination of x_2

$$\mathrm{x}_2 = \mathbf{A}_{22}^{-1}\left(\mathrm{b}_2 - \mathbf{A}_{21}\mathrm{x}_1\right)$$

$$\left(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)\mathrm{x}_1 = \mathrm{b}_1 - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathrm{b}_2$$

Schur Complement

$$\mathbf{S} \equiv \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}, \quad \mathbf{S}\mathrm{x}_2 = \mathrm{b}_1 - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathrm{b}_2$$

Change of Dependent Variables

$$\mathbf{V}'(x) = x\mathbf{J}(x)\mathbf{V}(x), \quad \det \mathbf{V}(x) \neq 0$$

$$\mathbf{V} = \mathbf{P}\mathbf{W}, \quad \det \mathbf{P} \neq 0, \quad \mathbf{W}' = x\mathbf{B}\mathbf{W}$$

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P} - x^{-1}\mathbf{P}^{-1}\mathbf{P}', \quad \mathbf{P}' = x\left(\mathbf{J}\mathbf{P} - \mathbf{P}\mathbf{B}\right)$$

$$\mathbf{J} = \sum_{k=0}^{\infty} \mathbf{J}_k x^{-2k}, \quad \mathbf{P} = \sum_{k=0}^{\infty} \mathbf{P}_k x^{-2k}, \quad \mathbf{B} = \sum_{k=0}^{\infty} \mathbf{B}_k x^{-2k}$$

$$\mathbf{J}_0\mathbf{P}_0 - \mathbf{P}_0\mathbf{B}_0 = 0$$

$$\mathbf{J}_0\mathbf{P}_k - \mathbf{P}_k\mathbf{B}_0 = \sum_{l=0}^{k-1} \left(\mathbf{P}_l\mathbf{B}_{k-l} - \mathbf{J}_{k-l}\mathbf{P}_l\right) - 2(k-1)\mathbf{P}_{k-1}$$

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Splitting Transformation:

Order-by-Order Determination of Schur Complement

$$\mathbf{J}_0 = \begin{pmatrix} \mathbf{J}_0^{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_0^{22} \end{pmatrix}, \quad \mathbf{B}_0 = \mathbf{J}_0, \quad \mathbf{P}_0 = \mathbf{I}$$

$$\mathbf{J}_0 \mathbf{P}_k - \mathbf{P}_k \mathbf{J}_0 = \mathbf{B}_k - \mathbf{K}_k$$

$$\mathbf{K}_k \equiv \mathbf{J}_k + 2(k-1)\mathbf{P}_{k-1} + \sum_{l=1}^{k-1} \left(\mathbf{J}_{k-l} \mathbf{P}_l - \mathbf{P}_l \mathbf{B}_{k-l} \right)$$

$$\mathbf{B}_k = \begin{pmatrix} \mathbf{B}_k^{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_k^{22} \end{pmatrix}, \quad \mathbf{P}_k = \begin{pmatrix} \mathbf{0} & \mathbf{P}_k^{12} \\ \mathbf{P}_k^{21} & \mathbf{0} \end{pmatrix}, \quad \mathbf{K}_k = \begin{pmatrix} \mathbf{K}_k^{11} & \mathbf{K}_k^{12} \\ \mathbf{K}_k^{21} & \mathbf{K}_k^{22} \end{pmatrix}, \quad k > 0$$

$$\begin{aligned} \mathbf{B}_k^{11} &= \mathbf{K}_k^{11} \\ \mathbf{B}_k^{22} &= \mathbf{K}_k^{22} \end{aligned}$$

$$\begin{aligned} \mathbf{J}_0^{22} \mathbf{P}_k^{21} - \mathbf{P}_k^{21} \mathbf{J}_0^{11} &= -\mathbf{K}_k^{21} \\ \mathbf{J}_0^{11} \mathbf{P}_k^{12} - \mathbf{P}_k^{12} \mathbf{J}_0^{22} &= -\mathbf{K}_k^{12} \end{aligned}$$

Equation for Power-Like Solutions

$$\mathbf{W}' = x\mathbf{B}\mathbf{W}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}^{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{22} \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} \mathbf{W}^{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{22} \end{pmatrix}$$

$$\mathbf{W}'^{11} = x\mathbf{B}^{11}\mathbf{W}^{11}, \quad \mathbf{B}^{11} = \mathbf{B}_0^{11} + x^{-2}\mathbf{B}_1^{11} + x^{-4}\mathbf{B}_2^{11} + \cdots$$

$$\mathbf{B}_0^{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B}_1^{11} = \begin{pmatrix} H & -KH^2Q^2 \\ 0 & 1-H \end{pmatrix}, \quad \mathbf{B}_2^{11} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{aligned}
b_{11} &= -HQ^2\left[1 + K\left(E + F - H + H^2\right)\right] \\
b_{12} &= HQ\left\{1 - HK\left[E - Q^3\left(1 + FK + H^2K\right)\right] + GH\left(1 + KQ^3\right)\right\} \\
b_{21} &= 2(H-1) - \left(E + F + H^2\right) \\
b_{22} &= Q^2\left\{-2 + H\left[1 + K\left(E + F - H + H^2\right)\right]\right\}
\end{aligned}$$

Power Series Expansion

$$\mathbf{W}' = x\mathbf{B}\mathbf{W}, \quad \mathbf{B} = \sum_{k=0}^{\infty} \mathbf{B}_k x^{-2k}$$

$$\begin{aligned}
\mathbf{W}(x) &= \sum_{k=0}^{\infty} \mathbf{W}_k x^{\mathbf{R}-2k\mathbf{I}} \\
x\mathbf{B}\mathbf{W} &= x \sum_{k=0}^{\infty} \sum_{l=0}^{k} \mathbf{B}_l \mathbf{W}_{k-l} x^{\mathbf{R}-2k\mathbf{I}} \\
\mathbf{W}'(x) &= x^{-1} \sum_{k=0}^{\infty} \mathbf{W}_k \left(\mathbf{R} - 2k\mathbf{I}\right) x^{\mathbf{R}-2k\mathbf{I}} \\
&= x \sum_{k=1}^{\infty} \mathbf{W}_{k-1} \left[\mathbf{R} - 2(k-1)\mathbf{I}\right] x^{\mathbf{R}-2k\mathbf{I}}
\end{aligned}$$

$$x \sum_{k=0}^{\infty} \left\{ \mathbf{B}_0 \mathbf{W}_k + \mathbf{B}_1 \mathbf{W}_{k-1} + \sum_{l=2}^{k} \mathbf{B}_l \mathbf{W}_{k-l} - \mathbf{W}_{k-1} \left[\mathbf{R} - 2(k-1)\mathbf{I}\right] \right\} x^{\mathbf{R}-2k\mathbf{I}} = \mathbf{0}$$

$$\mathbf{B}_0 \mathbf{W}_k = \mathbf{Y}_k \equiv \mathbf{W}_{k-1} \left[\mathbf{R} - 2(k-1)\mathbf{I}\right] - \mathbf{B}_1 \mathbf{W}_{k-1} - \sum_{l=2}^{k} \mathbf{B}_l \mathbf{W}_{k-l}$$

Lowest Order Solutions

Zeroth Order Operator

$$\mathbf{B}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{W}_k = \begin{pmatrix} w_k^{11} & w_k^{12} \\ w_k^{21} & w_k^{22} \end{pmatrix}$$

$$\mathbf{B}_0\mathbf{W}_k = \begin{pmatrix} w_k^{21} & w_k^{22} \\ 0 & 0 \end{pmatrix} = \mathbf{Y}_k = \begin{pmatrix} y_k^{11} & y_k^{12} \\ y_k^{21} & y_k^{22} \end{pmatrix}$$

$$w_k^{21} = y_k^{11}, \quad w_k^{22} = y_k^{12}, \quad y_k^{21} = y_k^{22} = 0$$

Zeroth Order Equation

$$\mathbf{B}_0\mathbf{W}_0 = \mathbf{0}, \quad \mathbf{B}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{W}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

First Order Equation

$$\mathbf{B}_0\mathbf{W}_1 = \mathbf{Y}_1 = \mathbf{W}_0\mathbf{R} - \mathbf{B}_1\mathbf{W}_0$$

$$\mathbf{R} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} H & -KH^2Q^2 \\ 0 & 1-H \end{pmatrix}$$

$$\begin{aligned} w_1^{21} &= y_1^{11} = r_1 - H \\ w_1^{22} &= y_1^{12} = r_2 - H \end{aligned}$$

Mercier Powers

Second Order Equation

$$\mathbf{B}_0\mathbf{W}_2 = \mathbf{Y}_2 = \mathbf{W}_1(\mathbf{R} - 2\mathbf{I}) - \mathbf{B}_1\mathbf{W}_1 - \mathbf{B}_2\mathbf{W}_0$$

$$\begin{aligned} y_2^{21} =& E + F + H + 2 - 3r_1 + r_1^2 = 0 \\ y_2^{22} =& E + F + H + 2 - 3r_2 + r_2^2 = 0 \end{aligned}$$

$$D_I = E + F + H - 1/4$$

$$\begin{aligned} r_1 &= 3/2 + \sqrt{-D_I} \\ r_2 &= 3/2 - \sqrt{-D_I} \end{aligned}$$

$$\begin{aligned} w_2^{21} = y_2^{11} = (r_1 - 2 - H)\, w_1^{11} + H^2KQ^2\,(r_1 - 1) + HQ^2\,[1 + K\,(E + F)] \\ w_2^{22} = y_2^{12} = (r_2 - 2 - H)\, w_1^{12} + H^2KQ^2\,(r_2 - 1) + HQ^2\,[1 + K\,(E + F)] \end{aligned}$$

Back Substitution

$$\begin{aligned} \mathbf{U} =& \mathbf{TPW} \\ =& \begin{pmatrix} \mathbf{T}^{11} & \mathbf{T}^{12} \\ \mathbf{T}^{21} & \mathbf{T}^{22} \end{pmatrix} \begin{pmatrix} \mathbf{P}^{11} & \mathbf{P}^{12} \\ \mathbf{P}^{21} & \mathbf{P}^{22} \end{pmatrix} \begin{pmatrix} \mathbf{W}^{11} \\ \mathbf{0} \end{pmatrix} \\ =& \begin{pmatrix} \mathbf{T}^{11} & \mathbf{T}^{12} \\ \mathbf{T}^{21} & \mathbf{T}^{22} \end{pmatrix} \begin{pmatrix} \mathbf{P}^{11} \\ \mathbf{P}^{21} \end{pmatrix} \mathbf{W}^{11} \\ =& \begin{pmatrix} \mathbf{T}^{11}\mathbf{P}^{11} + \mathbf{T}^{12}\mathbf{P}^{21} \\ \mathbf{T}^{21}\mathbf{P}^{11} + \mathbf{T}^{22}\mathbf{P}^{21} \end{pmatrix} \mathbf{W}^{11} \end{aligned}$$

$$\Xi = u_2 = \sqrt{Q}\,[(p_{61} - p_{41})\, w_1 + (p_{62} - p_{42})\, w_2] \sim x^{\mathbf{R} - 2\mathbf{I}}$$

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Higher Order Solutions

Higher Order Equations

$$\mathbf{Z}_k \equiv \sum_{l=2}^{k} \mathbf{B}_l \mathbf{W}_{k-l} = \mathbf{B}_2 \mathbf{W}_{k-2} + \mathbf{B}_3 \mathbf{W}_{k-3} + \cdots + \mathbf{B}_k \mathbf{W}_0$$

$$\mathbf{B}_0 \mathbf{W}_k = \mathbf{Y}_k \equiv \mathbf{W}_{k-1} \left[\mathbf{R} - 2(k-1)\mathbf{I} \right] - \mathbf{B}_1 \mathbf{W}_{k-1} - \mathbf{Z}_k$$

Higher Order Solutions

$$\begin{aligned} w_k^{21} &= y_k^{11} \\ w_k^{22} &= y_k^{12} \end{aligned}$$

$$\begin{aligned} (r_1 - 2k - 1 + H) w_k^{21} &= z_k^{21} \\ (r_2 - 2k - 1 + H) w_k^{22} &= z_k^{22} \end{aligned}$$

$$\begin{aligned} (r_1 - 2k - 1 + H) y_k^{11} &= z_k^{21} \\ (r_2 - 2k - 1 + H) y_k^{12} &= z_k^{22} \end{aligned}$$

$$\begin{aligned} z_k^{21} &= (r_1 - 2k - 1 + H) \left[(r_1 - 2k + 2 - H) w_{k-1}^{11} + H^2 K Q^2 w_{k-1}^{21} - z_k^{11} \right] \\ z_k^{22} &= (r_2 - 2k - 1 + H) \left[(r_2 - 2k + 2 - H) w_{k-1}^{12} + H^2 K Q^2 w_{k-1}^{22} - z_k^{12} \right] \end{aligned}$$

Status and Future Work

- A new power series solution has been derived for the large-x limit of the resistive inner region equations. It has the right power-like behavior to match onto the outer region solutions, but different high-order terms.
- Conjecture: the error in the previous solution is due to omission of coupling to the exponential solutions through the Schur complement.
- A new computer code is being developed to diagnose this solution, to determine whether is provides improved asymptotic convergence..
- The new solution will be incorporated into the matching code and used to determine complex growth rates and eigenfunctions.
- Verification against the straight-through MARS-F code will be redone to determine whether the new solutions eliminate the discrepancies.
- The results of this study will be applied to a new treatment of the linear neoclassical tearing mode.