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Courant Institute of
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Magneto-Fluid Dynamics Division

Abstracts of Sherwood Theoretical Meeting March 30-31, 1967

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Abstracts of

SHERWOOD THEORETICAL MEETING

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Program

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9 a.m. Welcome, K. O. Friedrichs
Director, Courant Institute

SESSION I--Chairman, J. Berkowitz

2 p.m. SESSION II--Chairman, A. A. Blank

2 p.m. SESSION IV--Chairman, G. Morikawa

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Drift Ballooning Modes in Multipoles

MARSHALL N. ROSENBLUTH and B. COPPI

University of California, San Diego
and General Atomic Division/General Dynamics Corporation

It is well known that in the presence of unfavorable curvature the drift mode is unstable even for zero Larmor radius. We consider an electrostatic perturbation of a multipole equilibrium with variable curvature in the limit that the frequency is high compared to the frequency of ion longitudinal motion but low compared to the electron frequencies. In this limit the electrons adjust to a Boltzmann distribution along the field lines while the ions move with the $\frac{E \times B}{B^2}$ drift. In addition an electron Landau damping will exist for slow particles satisfying $\omega = n\omega_f$ where ω_f is the frequency of the longitudinal motion. This will be destabilizing if $\omega < k a_i \frac{1}{n} \frac{dn}{dx} v_{thi} T_e/T_i$. For modes localized in the neighborhood of the unstable curvature this will be true. We find that the condition for the existence of such localized ballooning instabilities is given by

$$(k^2 a_i^2)\ \ell^2 \frac{1}{B}\frac{dB}{dr}\frac{1}{n}\frac{dn}{dr}\frac{T_e}{T_i} > 1$$

where ℓ is the length of the region of unfavorable curvature.

Low Frequency Limit of Interchange Stability

MARSHALL N. ROSENBLUTH

University of California, San Diego
and General Atomic Division/General Dynamics Corporation

For simplicity we consider an axisymmetric torus, e.g., a multipole with coordinates θ, Ψ the flux coordinate, and χ the magnetic potential. Consider an electrostatic perturbation $\phi \sim \phi(\chi, \Psi)\, e^{im\theta} e^{i\omega t}$ where ω is taken to be small compared to the frequencies characteristic of the guiding center motion, i.e., the bounce frequency for trapped particles and the frequency around the toroidal minor axis for untrapped particles. In this limit the particle sees only the time average of the potential over its orbit, and for a Maxwellian equilibrium the perturbed distribution is given by:

$$f_1(\chi, \Psi, E, \mu) = -\frac{f_o e\phi}{T} + f_o \frac{e}{T} \frac{\omega + \frac{mT}{e} \frac{1}{f_o} \frac{\partial f_o}{\partial \Psi}}{\omega + m\bar{v}_d} \bar{\phi}$$

where v_d is the usual guiding center drift $(d\theta/dt)$ and the bars indicate a time average over the unperturbed orbit. For untrapped particles in the presence of shear $\bar{\phi}$ vanishes.

Using quasineutrality we may derive a variational expression which is an extremal for ω^2 in the cases $\frac{1}{n}\frac{dn}{dx} \gg \frac{1}{B}\frac{dB}{dx}$ or $T_i = T_e$. A necessary and sufficient condition for stability is then that v_d must be favorable for all particles, or all trapped particles in the presence of shear, i.e., $\left.\frac{\partial J}{\partial \Psi}\right)_{E,\mu} \frac{\partial f}{\partial \Psi} < 0$. Growth rates are of order a_i/r smaller than hydrodynamic. Conventional multipoles violate this condition for barely trapped particles. However, we show that a proper minimum J geometry may be constructed by modification of the multipole field.

Guiding Center Stability

D. B. NELSON

Oak Ridge National Laboratory

Using a variational principle for the guiding center plasma we examined the conjectured theorem that at suitably low β, interchange stability plus local stability is sufficient for absolute stability. The geometry chosen for this examination is a mirror machine with insulated ends, so that the magnetic lines are free rather than tied at the ends.

Although we have not yet proven this theorem in full generality we have shown that for the subclass of equilibria which, roughly speaking, admit no neutral interchanges the theorem is true if the magnetic field and plasma pressure are smooth enough. The requirement of low β results from integrating the small pressure derivative in from the plasma boundaries. More exactly, if

$$\min_{U} \frac{\delta^2 \Phi(U)}{\int U^2 dv} > 0$$

where $\delta^2\Phi$ is the second variation and the minimum is taken over all interchanges, then the equilibrium is stable if it is locally stable and if $\nabla(p_\perp + B^2/2)$ and the tensors ∇B and $\partial^2(p_\perp + B^2/2)/\partial x_i \partial x_j$ are sufficiently small.

The method of proof uses the fact that any variation U can be decomposed into an interchange U^1 plus a variation U^o which vanishes on one end. Then the second variation, as a quadratic form, becomes

$$\delta^2\Phi(U) = Q(U^o) + Q(U^o,U^1) + Q(U^1)$$

Now $Q(U^1)$ is positive by hypothesis. We have shown that $Q(U^o)$ is positive, proving a theorem that for a smooth enough (or short enough) plasma with one end tied, local stability suffices for absolute stability. The work remaining to complete the proof involves bounding the cross terms $Q(U^o,U^1)$. We have already done this for the case given above.

Absolute Loss-Cone Instabilities*

R. A. DORY and G. E. GUEST

Oak Ridge National Laboratory

Rosenbluth and Post[1] first analyzed the high-frequency electrostatic instabilities resulting from inverted distribution of gyration energy caused by the loss-cone in mirror traps or by velocity-dependent loss mechanisms in other configurations. Because the modes were found to be convective in nature, it was possible to find heuristic sets of mirror trap parameters that provide adequate stability for fusion purposes. The inverted distributions also cause other instabilities which occur at lower density and which are associated with the gyration frequency of the ions. These were investigated earlier by the present authors,[2] but the propagation properties of the waves were not determined. We have extended the calculations and have found that under a range of circumstances, these modes are absolute rather than convective, so that the stabilization invoked for the Rosenbluth-Post modes does not apply here. We present a preliminary investigation of the parameter ranges for which the instabilities are absolute. It is found that even very smooth inverted distributions lead to absolute instabilities. To obtain a degree of stabilization it appears likely that finite length effects will be needed; either through restrictions on machine length so that wavelengths are bounded and Landau damping will be effective (electron heating may be necessary to provide sufficient damping), or if these length requirements are too stringent, it may become necessary to treat the finite boundary value problems in detail in order to find new stabilizing mechanisms.

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[*] Research sponsored by the U. S. Atomic Energy Commission under contract with the Union Carbide Corporation.

[1] M. N. Rosenbluth and R. F. Post, Phys. Fluids 8, 547 (1965).

[2] G. E. Guest and R. A. Dory, Phys. Fluids 8, 1853 (1965).

Flute-Like "Loss-Cone" Instabilities in Multi-Component Plasmas*

W. M. FARR and R. L. BUDWINE
Oak Ridge National Laboratory

Detailed calculations have been made to determine instability thresholds for high-frequency flute-like oscillations in plasmas containing two distinct groups of ions, one of which is relatively cold and well thermalized and the other of which is of the class of "loss-cone" distributions studied by Guest and Dory.[1] The results are displayed in the parameter space generated by N_C/N_H and ω^2_p/ω^2_c, where N_C and N_H are the cold- and hot-ion densities, ω_p is the plasma frequency corresponding to the total ion density, and ω_c is the ion gyrofrequency. These stability boundaries, corresponding to the "most unstable" ranges of wavelength, are compared to the less specific thresholds given earlier by Perlstein et al.[2]

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* Research sponsored by the U. S. Atomic Energy Commission under contract with the Union Carbide Corporation

[1] G. E. Guest and R. A. Dory, Phys. Fluids 8, 1853 (1965).

[2] L. D. Perlstein, M. N. Rosenbluth and D. B. Chang, Phys. Fluids 9, 953 (1966).

Scattering from Magnetic Fluctuations*

WILLIAM B. THOMPSON

University of California at San Diego
La Jolla, California

Scattering of optical and radio frequency radiation from density fluctuations in a plasma has been extensively studied and bids fair to become a useful diagnostic tool. Recent observations of Faraday rotation in the optical region imply that fluctuations in the magnetic field should give rise to a further scattering which might also prove useful in the study of turbulent plasmas.

The usual scattering results may be obtained from the expression for the scattered spectrum due to a distribution of polarization $\underset{\sim}{P}$ over a volume V_s distance $\underset{\sim}{R}$ from the point of observation

$$I(\omega)d\omega = \frac{\omega^4}{2\pi c^3}\frac{V_s}{R^2}\int d^3r'\left\langle \underset{\sim}{P}_\perp(r,\omega)\cdot \underset{\sim}{P}_\perp^x(o,\omega)\right\rangle e^{\frac{i\omega}{c}\hat{\underset{\sim}{R}}\cdot\underset{\sim}{r}}$$

where $\underset{\sim}{P}_\perp$ is the projection of $\underset{\sim}{P}$ normal to $\underset{\sim}{R}$.

One writes $\underset{\sim}{P} = -(\omega_p^2/\omega^2)E_o$ and $\omega_p^2 = \frac{4\pi e^2}{m}\,[n_o + \delta n(\underset{\sim}{x}\cdot t)]$ whence follows

$$\frac{I_s(\omega)}{I_o}\,d\omega = n_o V_s\left[8\pi[e^2/mc^2]^2\right] \;,$$

$$\langle(\delta n/n)^2\rangle\; f(\omega-\omega_o,\underset{\sim}{k}-\underset{\sim}{k}_o)\cdot(\hat{\underset{\sim}{E}}_o(k_o,\omega_o)\;-\;\hat{\underset{\sim}{E}}_o\cdot\hat{\underset{\sim}{R}})^2$$

where $\langle \delta n^2 \rangle f(\omega,k) = \int \langle \delta n(r,\omega)\delta n^x(o,\omega) \rangle e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} d^3r'd\omega$

so that f is a normalized correlation function.

In a magnetic field, B, the polatization may be written

$$\underset{\sim}{P} = - \omega_p^2/\omega^2 \; [\underset{\sim}{E} + \frac{\Omega}{\omega} \underset{\sim}{E} \times \underset{\sim}{b} + O(\frac{\Omega}{\omega})^2]$$

and in the presence of fluctuations as

$$\delta\underset{\sim}{P} = - \omega_p^2/\omega^2 \left\{ \frac{\delta n}{n} [\underset{\sim}{E} + \frac{\Omega}{\omega} \underset{\sim}{E} \times \underset{\sim}{b}] + \frac{\Omega}{\omega} [\frac{\delta B}{B} \underset{\sim}{E} \times \underset{\sim}{b} + \underset{\sim}{E} \times \delta\underset{\sim}{b}] \right.$$

where Ω is the gyro frequency in the field B, $\frac{eB}{mc}$ and $\underset{\sim}{b}$, the unit vector along $\underset{\sim}{B}$. Of the terms in here, the second gives a depolarized modification $\sim (\Omega/\omega)^2$ the usual scattered wave, and is perhaps negligible, and the second and third, which arise from magnetic fluctuations, are orthogonal to the first. This orthogonality persists in the scattered field only in special directions (e.g., backward, forward!) of scattering. In these special directions the field fluctuation effect may be separated from the rest by controlling the polarization of the observed wave. The added scattered intensities have the form (for back scattering)

$$I_B/I_o = nV_s \; 8\pi(e^2/mc^2)^2 \langle \delta B^2 \rangle / B^2 \; g(\omega-\omega_o, k-k_o)$$

$$I_\theta/I_o = nV_s \; 8\pi(e^2/mc^2)^2 \langle \delta\theta^2 \rangle \; h(\omega-\omega_o, k-k_o)$$

where ω, $\underset{\sim}{k}$, ω_o, $\underset{\sim}{k}_o$ = scattered and incident frequency and wave number

$$(g,h) = (\Omega/\omega)^2 n(\overline{g},\overline{h})$$

and

$$\langle \delta B^2 \rangle \overline{g} = \int \langle \delta B(r,\omega) \delta B(o,\omega) \rangle \; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \, d^3r$$

$$\langle \delta\theta^2 \rangle \overline{h} = \int \langle \hat{E}_o \times \delta\hat{h}(r,\omega) \cdot \hat{E}_o \times \delta\hat{h}(o,\omega) \quad e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \, d^3r$$

g and h have order of magnitude $(\Omega/\omega)^2 n\ell^3$ where ℓ is the correlation length and, although the frequency ratio is necessarily small, the factor $n\ell^3$ may be very large. Thus magnetic scattering may be as large as, or much larger than scattering from density fluctuations. It can be shown to be small in thermal equilibrium, and is probably small in low β plasmas; however, in high β turbulent plasmas it may be of considerable importance.

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*Work supported in part by the U. S. Atomic Energy Commission.

Bremsstrahlung Emission in Plasmas

ALAN OPPENHEIM

Polytechnic Institute of Brooklyn

We develop a formula for the power density of Bremsstrahlung radiation from binary interactions emitted from a fully ionized non-relativistic plasma. The calculation is strictly classical. The resulting formula contains a linear variation with temperature rather than the variation with the square root of the temperature as computed from Kramer's results. The two formulas are numerically equal at $4700^{\circ}K$. Above this temperature the linear formula clearly predicts a higher rate of radiation emission.

Boundary Value Problem for Bernstein Resonances

HAROLD WEITZNER

New York University
Courant Institute of Mathematical Sciences

A plasma varying with only one spatial coordinate described by the linearized Vlasov equation with an unperturbed magnetic field perpendicular to the direction of spatial variation is considered. The electric field, with only the longitudinal part included, is given by Poisson's equation. A method of solution is described for the mixed initial value-boundary value problem for the case of a plasma bounded by one wall only. When the plasma is driven by a harmonically oscillating electric field at the wall, the behavior of the electric field for large time is examined. The initial data also contribute to the long-time limit of the electric field, and this part is given. The limiting value of the electric field for large separation from the wall is described as a function of driving frequency. The line shape as a function of frequency is given near resonance, and the full time behavior at resonance is examined.

Echo Phenomena Associated with Landau Damping*

T. M. O'NEIL

General Atomic Division
General Dynamics Corporation

R. W. GOULD

California Institute of Technology

A calculation is presented which predicts the following echo phenomena. A wave of wave length k_1 is excited in a plasma and then Landau damps away; after a time T, a wave of wave length k_2 is excited in the plasma and it also Landau damps away; then after a time $t = T[k_2/(k_2-k_1)]$, a third wave (i.e., the echo) spontaneously appears in the plasma.

The basic mechanism behind these phenomena can be easily understood. When the first wave damps away, it leaves a perturbation in the particle distribution function of the form $f_{k_1}(v)\cos(k_1x - k_1vt)$. There is no electric field associated with this perturbation, since for large t the perturbation is a rapidly oscillating function of v, and a velocity integral over the perturbation (i.e., the charge density) will phase mix to zero. The second wave modulates the unperturbed particle distribution and leaves a similar perturbation, but it also modulates the perturbation due to the first wave, leaving a second order perturbation of the form $f_{k_1}(v)f_{k_2}(v)\cos[k_1x - k_1vt]\cos[k_2x - k_2v(t-T)$

This perturbation can be rewritten as

$f_{k_1}(v)f_{k_2}(v)\ \cos[(k_1+k_2)x - (k_1+k_2)vt + k_2vT] +$
$\cos[(k_1-k_2)x + (k_2-k_1)vt - k_2vT]$. In the second cosine, the coefficient of v will vanish when $t = T[k_2(k_2-k_1)]$; so at this time the second term will no longer be a rapidly oscillating function of v. Consequently, a velocity integral over this term will not phase mix to zero, and an electric field will reappear in the plasma. Of course, this qualitative explanation has been verified by a perturbation solution of the Vlasov equation.

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* This research was sponsored by the Defense Support Agency under contract DA 49-146-XZ-486.

Computation of Equilibrium Configurations of a Toroidal Plasma

S. FISHER and J. KILLEEN

Lawrence Radiation Laboratory, University of California
Livermore, California

We have developed a Fortran program for computing static hydromagnetic equilibria for a toroidal plasma with scalar pressure. We assume that the system is axially symmetric so we can apply these calculations to such configurations as the levitron or floating ring devices. Grad and Rubin[1] have shown that a stream function $\psi(r,z)$ can be introduced which satisfies the equation

$$r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial z^2} + ff' + r^2 g' = 0$$

where $f(\psi) = p$ and $g(\psi) = rB_{\phi}$, i.e., the pressure and azimuthal component of the field are functions of ψ only. This equation was solved analytically[2] for a special form of f and g. We solve the above equation by finite-difference methods using an alternating direction implicit method developed for an earlier problem with the same equation.[3] The choice of f and g is quite flexible and various profiles can be studied. We follow the procedure of the earlier problem,[3] and let $\psi = \psi_c + \psi_p$, where ψ_c is the stream function of the vacuum magnetic field and ψ_p is the stream function of the magnetic field due to the plasma. The function

$\psi_c(r,z)$ is then given by the experimental configuration and is computed from the coil elements.[4] We choose fields with closed constant-ψ_c surfaces and assume that the plasma is contained between two such surfaces. The solution for $\psi_p(r,z)$ is by iteration of the difference equation, and a succession of equilibrium solutions can be found for increasing values of the pressure maximum.

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[*] Work performed under the auspices of the U.S. Atomic Energy Commission.

[1] H. Grad and H. Rubin, Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, 31, 190 (1958), United Nations, N.Y.

[2] E. W. Laing, S. J. Roberts and R. T. P. Whipple, J. Nucl. Energy, Part C 1, 49 (1959).

[3] J. Killeen and K.J. Whiteman, Phys. Fluids 9, 1846 (1966).

[4] J. Killeen, H. P. Furth, R. P. Freis, "Calculation of Toroidal Magnetic Field Configurations", UCRL-50161, January 1967.

Toroidal Magnetic Fields with a Circular Magnetic Axis

GEORG KNORR

Department of Physics
University of California, Los Angeles

ECKHARD REBHAN

Institute for Plasma Physics, Garching, Germany

The general solution of Laplace's equation for vacuum magnetic fields generated by arbitrary surface currents on a circular toroid is given in toroidal coordinates. Magnetic surfaces are constructed around a circular axis as an expansion in powers of the aspect ratio. For a simple model this expansion can be extended in principle to arbitrarily high orders. Whenever the rotational transform $\iota/2\pi$ on the axis assumes rational values m/n ($m < n$ and without common divider) there might, but must not, appear resonances (e.g., by stray fields with periodicity n) which lead to a destruction of the surfaces. The values of V'' and $\iota/2\pi$ are computed on the axis and expressed by parameters of the field. On the other hand, the field generating surface currents can also be expressed by the field parameters thus relating favorable field properties directly to the currents. It is shown that fields with arbitrarily deep magnetic wells and large ι (and shear) can be produced by relatively simple helical coils.

Toroidal Magnetic Well Configurations

HAROLD GRAD

New York University
Courant Institute of Mathematical Sciences

We analyze the following statement:

"A plasma is diamagnetic. It is therefore stable in a magnetic well. A magnetic well is topologically impossible in a torus. We must therefore settle for an average well to obtain stability in a torus."

1. Minimum B (as it is interpreted in a mirror machine) is not related to plasma stability.

2. Minimum average B ($\int d\ell/B$, V", etc.) has almost no relation to stability.

3. Minimum average B is not a weakened form of minimum B; they are essentially unrelated.

4. There are simple examples of a plasma which is paramagnetic. Whether it is paramagnetic or diamagnetic has no relation to its stability either in a well or not.

5. There are very simple toroidal vacuum magnetic wells. When combined with a proper distribution function [Taylor's $f(\varepsilon,\mu)$] they are stable ($\delta W > 0$). No considerations of shear, line curvature, or V" enter.

Equilibrium Electric Fields in Multipoles*

T. K. FOWLER

General Atomic Division/General Dynamics Corporation

Low-β equilibrium configurations in symmetric toroidal magnetic geometries, such as multipoles, are examined when toroidal effects are important. In this regime, locally Maxwellian equilibrium distributions require an equilibrium electric field. For $T_e \ll T_i$, the extreme cases are: (1) ions held electrostatically with potential $e\phi = T_i$, and (2) ions held magnetically with potential $e\phi = (R_i^2/R_p R_M)T_e$ where R_i is the ion gyroradius and R_p and R_M are the minor and major radii. The first case is characterized by zero ion current, small total angular momentum, uniform pressure along magnetic lines and $\vec{E}\cdot\vec{B} = 0$. In the second case, ions carry current and a sizeable momentum, and there must be a small electric field along $\vec{B}$ which pushes electrons toward outer radii where ions are driven by centrifugal forces (the toroidal effect). Some implications of these results will be discussed.

*This work was carried out under a joint General Atomic-Texas Atomic Energy Research Foundation program on controlled thermonuclear research.

Closed Line Toroidal Magnetic Field Configurations*

H. P. FURTH+ and J. KILLEEN

Lawrence Radiation Laboratory, University of California
Livermore, California

There is some interest in toroidal magnetic field configurations in which the field lines close upon themselves and have a maximum of $\oint d\ell/B$ in the plasma region. Floating ring devices have this property. A linear periodic system with this property, but without a floating conductor was discussed by Furth and Rosenbluth.[1] Recently a toroidal configuration with this property was studied by Taylor.[2] We have also been studying toroidal configurations without floating conductors. To do this we have found it useful to use a spherical coordinate system ρ, θ, ϕ. The magnetic field consists of two parts, the ordinary toroidal field given by $B_\phi = \frac{\alpha}{\rho \sin\theta}$ and a field $\vec{B} = \nabla\chi$, where the function $\chi(\rho,\theta,\phi)$ is a sum of various spherical harmonics. Approximate analytical work has indicated that a stable $\oint d\ell/B$ region can be obtained. To study this exactly we have developed a new Fortran program called TUBE V which solves for the field line trajectories in a torus, but in spherical coordinates. This technique of employing spherical harmonics is useful in studying more general stellerator fields. We shall present results of some closed line configurations.

*Work performed under the auspices of the U.S. Atomic Energy Commission.

+Now at Plasma Physics Laboratory, Princeton University, P. O. Box 451, Princeton, New Jersey.

[1] H. P. Furth and M. N. Rosenbluth, Phys. Fluids 7, 764 (1964).

[2] J. B. Taylor, "Anti-Symmetric Toroidal Containment Systems," Preprint Culham Laboratory, 1966.

Diffusion in a Cylindrical Force Free Field

M. BINEAU

New York University

Courant Institute of Mathematical Sciences

The choice of an arbitrary function $f(r)$ in an internal $(0,R)$ or (R_1,R_2) as the ratio of the current density to the magnetic field determines a unique cylindrically symmetric force free field in various situations. This occurs for a small ε, where ε measures the ratio of the magnetic field induced by the current to the given vacuum field. This occurs also for arbitrary ε and appropriate boundary conditions leading to a Volterra equation. The medium supporting the force free field is assumed to be resistive and its behavior governed by Ohm's law, the force free condition and Maxwell's equations. When the externally induced part of the field is kept constant, the velocity v of the medium, which is also the drift velocity of individual particles $v = \frac{E \times B}{B^2}$, is obtained explicitly after solving equations in integral form for the time derivative of the current density. For small ε the solution can be obtained as an expansion in powers of ε.

The lowest order term in ε is

$$v = \varepsilon\eta r \int_r^R \frac{f(x)}{x} \left(f(x) - \frac{2}{x^2} \int_0^x t\, f(t)\, dt\right) dx \quad \text{for}$$

a force free field in a domain (O,R) and resistivity η. For a force free field in a domain (R_1,R_2) with axis Oz (hard core configuragion), two natural boundary conditions can be considered, namely $\frac{\partial B_z(R_1)}{\partial t} = 0$ or $\frac{\partial B_z(R_2)}{\partial t} = 0$. The corresponding velocities are to lowest order in ε,

$$v = \eta \frac{r^2 B_\theta}{B_z} \int_{R_1}^{r} \frac{2f(x)\, dx}{x^3\left(1+\frac{B_\theta^2}{B_z^2}\right)} \quad \text{and} \quad v = -\eta \frac{r^2 B_\theta}{B_z} \int_{r}^{R_2} \frac{2f(x)dx}{x^3\left(1+\frac{B_\theta^2}{B_z^2}\right)} .$$

These formulae may be compared with the relative velocity $v - v_M = -\eta f \frac{B_\theta}{B_z}$, where v_M is the velocity $\frac{dR}{dt}$ which preserves constant flux in a circle of radius R. Note that the velocity is going into the domain if the current in the medium flows in the opposite direction to the current in the hard core.

Energy Variational Principles for Steady Hydromagnetic Flows in a Toroid

M. D. KRUSKAL and C. H. SU

Plasma Physics Laboratory, Princeton University

We consider ideal hydromagnetic flows in a toroid. The work is an extension of Kruskal and Kulsrud's [1] study on static equilibria. It is shown here that a hydromagnetic flow is steady if and only if it makes the total energy of the system variationally stationary under the constraint that the five surface functions not vary:

$$s(m)\ ,\ \psi(m)\ ,\quad (m)\ ,\ W(m)\ ,\ N(m)$$

representing specific entropy, long- and short-way magnetic fluxes, and two constants of motion pertinent to flow velocity. Here $m = M(x)$ is the magnetic surface enclosing mass m. A detailed discussion on the last two constants of motion $W(m)$ and $N(m)$ is given based on recent work of Newcomb [2]. Two distinct energy variational principles are constructed. Finally it is shown that one can derive from our energy variational principle the Lagrangian variational principle of Greene and Karlson [3].

[1] M.D. Kruskal and R.M. Kulsrud, Phys. Fluids 1, 265 (1958).

[2] W.A. Newcomb, UCRL-14884, University of California, Livermore, California (1966).

[3]J. M. Greene and E. T. Karlson, Princeton Plasma Physics Laboratory Report MATT-478 (1966).

Acceleration of Macroparticles to Very High Velocities by Megagauss Fields

F. WINTERBERG

Desert Research Institute and Department of Physics
University of Nevada, Reno, Nevada

A new, very promising method for the acceleration of macroparticles to extremely high velocities by megagauss fields is proposed. By this method, it seems to be possible to reach velocities of the order of 10^7 cm/sec and perhaps 10^8 cm/sec. It was previously shown by the author[1] that a small thermonuclear explosion can be triggered by the impact of macroparticles with velocities exceeding 10^7 cm/sec in dense T-D. The described new method of macroparticle acceleration may therefore lead to the controlled release of thermonuclear energy. This method of acceleration is based on a new and very promising method for the generation of magagauss fields. In contrast to previous methods using high explosives, the new method to generate megagauss fields employs a fast moving projectile for flux compression. The projectile is fired either by an ordinary or ultimately by a light gas gun.

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[1] Case Institute of Technology, Plasma Research Program, Technical Report A21, June 1963 and Z. Naturf. 19a, 231 (1964).

On-Line Solution of Loss-Cone Dispersion Relations

BURTON D. FRIED

University of California, Los Angeles and TRW Systems

JAMES E. McCUNE

Massachusetts Institute of Technology

A convenient technique for solving dispersion equations of the kind encountered in plasma physics has been described previously.[1] Application of this method to the complicated dispersion equations associated with a collisionless, magnetized plasma is greatly facilitated by an on-line digital computer which provides easy access to functional operations in the complex plane, as well as immediate displays of arcs or contours.[2] New features and recent refinements of this method are discussed and the technique is then applied to the case of the Rosenbluth-Post "loss cone" instability in its most virulent form, i.e., when the distribution function is of the form $\exp(-v_\parallel^2/a_\parallel^2)\delta(v_\perp - a_\perp)$. The transition from the case of small $x \equiv (k_\perp a_\perp/\omega_{ci})$, where a few terms of the Bessel function series dominate the behavior, to the case of large x is illustrated. The relation of these exact solutions to those obtained in the high density limiting case studied by Rosenbluth and Post[3] is discussed.

[1] J.E. McCune, Phys. Fluids 9, 2082 (1966) and B.D. Fried, "On-Line Root Finding in the Complex Plane," Report No. 9990-7308-R0000 TRW Systems, Redondo Beach, Calif.

[2]B.D. Fried, "Solving Mathematical Problems," Chap. VI of "On-line Computing", ed. by W. Karplus, McGraw Hill, New York, 1967. Also G.J. Culler, "Users Manual for an On-line System," Appendix, _loc. cit._

[3]M.N. Rosenbluth and R.F. Post, Phys Fluids 8, 547 (1965) and 9, 730 (1966).

Computer Simulation of Toroidal Plasma Confinement Including Dissipative and Inertial Effects

N. K. WINSOR, J. L. JOHNSON,* and J. M. DAWSON

Plasma Physics Laboratory, Princeton University

A computer program is being developed to study the confinement of low-β plasmas from the fluid point of view in simple axially symmetric configurations. It should be possible to study ideal stellarators (the model of Knorr [1] and Karlson [2]), toroidal pinches, and the levitron on the present model. With minor modifications it will be possible to study spherators and multipoles. The fluid equations adapted may include resistivity, viscous and gyro-viscous effects (general stress tensor), the Hall term, gradients in the pressure along the lines of force, and the effects of plasma inertia. This fluid approach should supplement the guiding center calculations of Bishop and Smith [3] who must solve a self-consistent problem. It should lead to a better understanding of the limitations of the present theories of plasma equilibria and the plasma losses associated with classical diffusion. It should help to clarify the role played by the mass motions of the plasma (streaming along field lines and plasma rotation) and the associated dissipative effects. The method for attacking this problem and the present state of the analysis will be presented.

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* On loan from Westinghouse Research Laboratories.

[1] G. Knorr, Phys. Fluids 8, 1334 (1965).

[2] E. T. Karlson, Princeton Plasma Physics Laboratory Report MATT-470 (1966).

[3] A. S. Bishop and C. G. Smith, Phys. Fluids 9, 1380 (1966).

Nonlinear Study of Vlasov's Equation for a Special Class of Distribution Functions

HERBERT L. BERK and KEITH V. ROBERTS
University of California, San Diego

Calculations on the "water bag model" of a two-stream instability have been made by a new numerical method. The distribution function is locally randomized but large scale nonlinear waves are observed. These waves are due to the creation of holes in phase space that formally behave as particles of negative mass since holes of like charge are attractive. On a longer time scale the system is expected to approach a Fermi distribution.

Fourier Hermite Expansion of the Vlasov Equation

M. R. FEIX and F. C. GRANT

The College of William and Mary and NASA Langley

The analytical and numerical properties of the double Fourier Hermite expansion of the Vlasov equation are examined in the linear and the nonlinear case. The difficulties associated with the closure conditions and the appearance of the microstructure in velocity space are shown to be rejected at larger and larger time where N (number of Hermite polynomials retained) increases. Moreover it is shown how in the linear problem the two limits $N \rightarrow \infty$, $\nu \rightarrow 0$ with $N\nu > 1$ or $N\nu < 1$ correspond respectively to the Landau and Van Kampen results when we introduce a small but finite Fokker Planck term. The Landau limit ($N\nu > 1$ $N \rightarrow \infty$ $\nu \rightarrow 0$) is much more interesting from a numerical point of view when nonlinear numerical calculations are considered.

In the nonlinear case the Fourier Hermite expansion is shown to correspond to a double quasi-linear, long-wavelength expansion. Numerical results are presented in two cases.

(A) Long wavelengths (kD = .05 and harmonics), α = .02 (depth of the initial density modulation). The small thermal effect destroys the exact periodicity of the cold plasma case, and the harmonics (but not the fundamental) display strong nonlinear behavior. Although the kD are very small, rather large number of Hermite polynomials (30) are necessary to

follow the system during long time ($\omega_p t = 80$), indicating a level of kineticity much stronger in the nonlinear case than in the linear one.

(B) Kinetic regime (kD = .5 and harmonics), the Landau damping is found to decrease when the amplitude of the field is increased in agreement with results obtained by Knorr[1] and Armstrong[2]. Numerical results indicate that this effect is a quasi-linear one (i.e., does not involve the first harmonic). A discussion of the role of this harmonic is presented for the damping case showing that although wave-wave interactions are, a priori, an effect of the same order as the quasi-linear one, it seems reasonable to disregard it for a small damping in agreement with the numerical results.

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[1] G. Knorr, Zeits. für Naturforschung 18a, 1304 (1963).

[2] Armstrong, APS meeting, Division of Plasma Physics, Boston, 1966, paper 7P4.

Self-Consistent Two-Dimensional Equilibrium Calculations Including Particle Loss

J. FREIDBERG and W. GROSSMANN

New York University
Courant Institute of Mathematical Sciences

We discuss computations of two-dimensional equilibria applicable for both low and high β mirror devices. Our aim is to study the effect of self mirroring due to particle currents. Specifically, we are interested in determining the self-consistent particle loss, taking into account self mirroring, and its effect on equilibrium flux surfaces. Our approach consists of calculating a current from an equilibrium solution of the Vlasov equation. This current is then used with Ampere's law to compute the flux surfaces.

More on Toroidal Geometry with Relativistic Electron Coils

N. C. CHRISTOFILOS

Lawrence Radiation Laboratory, University of California
Livermore, California

The combination of toroidal geometry with true magnetic well created by relativistic electron coils has been further studied. The required strength of the electron coils to satisfy T. K. Fowler's sufficient condition for hydromagnetic stability has been previously calculated.[1]

It has recently been observed that Fowler's depth of the well $(\Delta B/B)$ is related to the quantity (ω_b^2/ω_c^2) of the relativistic electron coils with the simple inequality

$$(\Delta B/B) \geq (\omega_b^2/\omega_c^2)$$

Hence the sufficient condition for hydromagnetic stability becomes

$$(\omega_b^2/\omega_c^2) = \frac{1}{2}\,\frac{T_e}{T_e + T_i}$$

where T_e, T_i is the electron and ion temperature respectively of the plasma confined in the magnetic well. This result indicates that the value of (ω_b^2/ω_c^2) is independent of the shape of the magnetic well created by the relativistic electron coils. For $T_e = T_i$, $(\omega_b^2/\omega_c^2) = .25$, a value which has been achieved in the Astron experimentally. At this level the E-layer is still free of instabilities. Thus a toroidal geometry of closed magnetic lines with true magnetic well created by a large number of E-layers disposed in a toroidal geometry is now in the realm of reality. A preliminary set of parameters for such a geometry will be presented.

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[*] Work performed under the auspices of the U.S. Atomic Energy Commission.

[1] H. Christofilos, The Physics of Fluids 9, 1425 (1966).

Interpretation of Experiments on Collisional Drift Modes

B. COPPI* and F. PERKINS

Plasma Physics Laboratory, Princeton University

We report the interpretation of experiments on low-temperature alkali plasmas in strong magnetic fields in terms of collisional drift modes, in which diffusion over the transverse wavelength resulting from ion-ion collisions, plays an important role. The experimental results agree with the theoretical predictions.

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* On leave from the University of California at San Diego.

The Inertial Confinement of Fusion Gases:

Theoretical Aspects

ROBERT L. HIRSCH

ITT Farnsworth Research Corporation,
Fort Wayne, Indiana

A theory is presented for the non-magnetic inertial confinement of fusion gases in spherical geometry. Assuming monoenergetic ion and electron distribution functions, the Poisson equation for bipolar charges is solved numerically. The results indicate that within a broad range of currents and energies periodic solutions are obtainable. These solutions are physically interpreted as the alternate formation of virtual anodes and virtual cathodes near the center of a hollow spherical cathode. This arrangement involves radial ion focusing, thereby creating dense regions of high kinetic energy ions near the center of the sphere, i.e., conditions favorable to a high fusion rate. "Real gas" modifications to the theory are discussed. In particular, a mechanism by which high fusion rates enhance ion trapping is presented. A means of direct energy conversion inherent to the concept is also described. A companion article describes encouraging results of recent experiments utilizing this concept.

Interaction Between Collisionless Shock and

Ion Acoustic Waves in a Magnetic Field*

N. A. KRALL and D. L. BOOK

General Atomic Division of General Dynamics Corporation
John Jay Hopkins Laboratory for Pure and Applied Science
San Diego, California

A recently proposed model to explain the behavior of turbulent heating experiments[1] by means of a spectrum of ion sound waves generated by diamagnetic drifts at the shock front is examined. The Vlasov equation is used to obtain the conditions for the existence of instability, and a quasi-linear equation describing the effect of the unstable spectrum on the particle distribution functions is derived. Non-linear corrections are studied and shown not to modify the picture significantly. It is concluded that the unstable spectrum is not built up to a level high enough to provide a mechanism for the particle heating and shock form observed.

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* This work was sponsored by the Defense Atomic Support Agency under contract DA-49-146-XZ-534.

[1] R. K. Kurtmullaev, Yu. E. Nesterikhin, V. I. Pilsky and R. Z. Sagdeev, "Plasma Heating by Collisionless Shock Waves," Plasma Physics and Controlled Nuclear Fusion Research, Vol. II, IAEA, Vienna, 367-387 (1966).

The Final Solution of the Problem of Non-linear Interactions*

R. BUDWINE**, O. C. ELDRIDGE†, E. G. HARRIS†, R. SUGIHARA†, G. M. WALTERS††

In a plasma there is a finite, and usually small, number of kinds of particles ond quasi-particles. The interaction among these may be described by an interaction Hamiltonian. Once this is known the problem of non-linear interactions is essentially solved; all that remains is the calculation of the development of processes of physical interest.

For instance, if $B_o = 0$, the particles are electrons and ions, and the quasi-particles are photons, plasmons and phonons. The coulomb interaction between particles and the photon-particle interactions must be regarded as fundamental and the others (such as particle-plasmon, plasmon-phonon, etc.) should be derived from them. We shall discuss our present knowledge of these interactions and the application to derivation of the quasi-linear equations and higher approximations, photon-photon scattering, radiation from plasma oscillations and acoustic waves, etc., etc.

If $B_o \neq 0$, there are several additional quasi-particles such as the quanta of Alfven waves, whistlers, etc. We have derived the particle-plasmon and plasmon-plasmon interaction Hamiltonians and used them in the derivation of quasi-linear equations and the wave-wave coupling corrections. Also, the interaction Hamiltonian between relativistic particles and (approximately) transverse waves has been derived and used to discuss synchrotron radiation and the cyclotron maser.

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** Oak Ridge National Laboratory.

† University of Tennessee and Oak Ridge National Laboratory.

†† University of Tennessee.

Non-Expandable, Non-Linear Effects

In Weakly Turbulent Plasmas

R. E. AAMODT

Department of Physics, University of Texas
Austin, Texas

The response of a homogeneous plasma containing a given spectrum of longitudinal waves to an infinitesimal test wave is found in a local approximation. The smearing of the wave-particle resonance and the resulting change of the dispersion relation for a gentle bump instability is presented. The time development of a weak homogeneous beam in this system is examined and some new non-linear effects important in beam stabilization are found.

Nonlinear Interaction of Positive and Negative Energy Modes in a Plasma*

B. COPPI†

Plasma Physics Laboratory, Princeton University

M. N. ROSENBLUTH

University of California at San Diego

R. N. SUDAN‡

Plasma Physics Laboratory, Princeton University

Nonlinear instabilities result from the interaction of positive and negative energy modes. In particular, a high-temperature plasma confined by magnetic mirrors is considered. For frequencies small compared to the ion gyrofrequency, in the linear regime two flute modes are associated with the magnetic field curvature and the density gradient. When the effect of the ion-diamagnetic velocity is considered both these modes are stabilized, but one is a positive energy mode and the other a negative energy mode if the magnetic curvature is unfavorable to stability. In the nonlinear regime, mode-mode coupling introduces an instability which upsets this finite Larmor radius stabilization. We also consider flute

modes at the harmonics of the ion gyrofrequency determined by the shape of the distribution function and the radial density gradient. For loss-cone type distributions, we observe that the transition from positive to negative energy modes occurs for wavelengths of the order of the ion gyroradius. For Maxwellian distributions the same transition occurs at much shorter wavelengths. Therefore, for a loss-cone distribution the relevant nonlinear instability is likely to cause a larger particle diffusion because longer wavelengths are involved. From a different point of view, this instability could well explain the rf bursts at harmonics of the gyrofrequency observed in several mirror experiments, because the non-convective nature of the relevant modes may allow them a sufficient time for interaction.

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†On leave from the University of California at San Diego.

‡On leave from Cornell University, Ithaca, New York

Development of a Large Amplitude Wave from the Two-Stream Instability

JAMES E. DRUMMOND

Boeing Scientific Research Laboratories
Seattle, Washington

The large amplitude steady state waves found by Sen are re-calculated and found to result from distributed charges, not concentrated dipoles. These solutions are then perturbed by a small growth rate and found to develop from lower amplitude waves. The steady state solutions are stable.

Quasi-Linear Theory for Alfven Waves

FANG C. HOH

Boeing Scientific Research Laboratories
Seattle, Washington

We consider the following equilibrium: An infinite homogeneous plasma is immersed in an uniform magnetic field B directed along the z axis. The plasma is collision free and has arbitrary pressure. The equilibrium is perturbed by a plane Alfvèn wave propagating along the z axis with wavenumber k. The Vlasov-Maxwell equations are used and are Fourier decomposed. According to the quasi-linear approximation, we only keep the non-linear terms in the diffusion equation that describes the time change of the Fourier component of the background distribution function $f_{k=o}$ or f_o.

The independent variables v_z and $v_\perp$, the particle velocities in the direction of and perpendicular to the magnetic field respectively are transformed into a new pair of variables; $\eta = v_\perp$ and $\xi = v_\perp^2 + v_z^2 + 2v_z v_A$, where v_A is the Alfven speed. The diffusion equation then reads:

$$\frac{\partial f_o}{\partial t} = \frac{\left(v_A^2 + \xi - \eta\right)^{\frac{1}{2}}}{v_A} \; \frac{1}{\eta} \frac{\partial}{\partial \eta} D(\xi, \eta, t) \frac{\partial f_o}{\partial \eta}$$

where D is a diffusion coefficient containing the square of the wave amplitude. The characteristic line of this parabolic differential equation is ξ = const. which describes an ellips in the $(v_\perp, v_z)$ plane. The unstable particles diffuse along the contours of the ellipses until f_o is constant along it, i.e., $\partial f_o/\partial\eta = 0$.

If we shoot a weak beam of plasma having velocities close to $v_\perp$ and $v_z \approx (\Omega - \omega)/k$ (Ω = ion cyclotron frequency and $\omega = kv_A$) into a Maxwellian plasma (a stable equilibrium), the beam ions will move along the ξ = const. contours and thereby lose their $v_\perp$ energy which partly goes to increase their v_z energy and partly to excite an Alfvèn wave.

Expansion of a Resistive Plasmoid in a Magnetic Field*

D. K. BHADRA

General Atomic Division of General Dynamics Corporation
John Jay Hopkins Laboratory for Pure and Applied Science
San Diego, California

Calculations are presented which show the effect of resistivity in the expansion of a laser produced hot plasma against a magnetic field. Without resistivity, the expanding plasma has periodic behavior due to bouncing back from the field; but with resistivity, there is superimposed a slow diffusion (across the field) on the periodic oscillations. This is illustrated by considering a spherically symmetric plasma produced by a giant laser pulse (delivering power of the order of 10^{10}W) illuminating a perticle of dimension 10^{-2}cm and subsequently following the time development of its radius, skin-depth and temperature by integrating the non-linear equations of motion and the Ohm's Law.

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Kelvin-Helmholtz Instability in Magneto-Plasmas

GEORGE SCHMIDT

Stevens Institute of Technology

A general equation describing instabilities in a low β magneto plasma under the influence of nonuniform electric field and gravitational field, both perpendicular to B_o, has recently been derived by Stringer and Schmidt.[1] This eigenvalue equation that contains finite gyro-radius effects has been shown to reduce in the appropriate limits into the equations obtained by Rosenbluth and Simon[2] for high density and low density plasmas. It has also been shown that in the very high density and zero gyro-radius limit the equation reduces to a known equation in hydrodynamics that describes the Kelvin-Helmholtz instability.

We have reduced[3] the high density equation by using a small (but not zero) gyro-radius ordering to a similar fluid equation and obtained necessary conditions for instability in terms of a modified Richardson number. In the case of no gravity a sufficient condition for instability has also been found. Note that the shear flow produced by the nonuniform electric field may induce instability even in the presence of stabilizing gravity.

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3. G. Emmert and G. Schmidt, to be published.

Normal Modes and Space-Time Correlations

J. K. PERCUS

New York University
Courant Institute of Mathematical Sciences

G. J. YEVICK

Stevens Institute of Technology

The normal mode analysis of space-time correlations in a many-body system requires a mechanism for decay, which may be attributed explicitly to velocity or current density fluctuations. The fashion in which these may be handled in an ensemble average of the microscopic hydrodynamic equations is not a priori transparent. This problem has been investigated by a step-by-step comparison with the readily obtained solution of the linearized microscopic Vlasov equation. The advantage of this framework is that no specific approximation as to the correlation structure has to be imposed.

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