The flux coordinate independent approach to plasma turbulence simulations

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Outline

- Motivation: plasma anisotropy and turbulence codes
- The flux coordinate independent (FCI) method to deal with plasma anisotropy
- Relation to various field-alignment methods
- Applications and tests:
  with the FENICIA finite difference code
  with the GYSELA semi-Lagrangian code
- Conclusions

The computational problem

Global, uniform grid, machine like ITER $a = 2m$, $R = 6m$

- Resolve the ion Larmor radius with four grid points, 1mm grid spacing
- Poloidal plane : $N_R \times N_Z = 4000 \times 4000$ points
- Toroidal direction 36000 points

$$N_{\rm points} \sim \rho_*^{-3} \sim 6 \times 10^{11}, \textbf{ unaffordable}$$

If one could work with a fixed (ρ_*-independent) number of toroidal points:

- $N_R = N_Z = 4000$, as before
- Perhaps $N_\phi = 64$

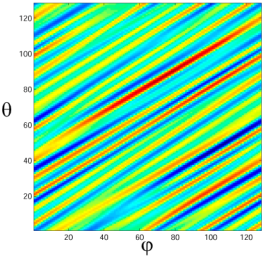
$$N_{\rm points} \sim \rho_*^{-2} \sim 10^9, \textbf{ feasible}$$

Achievable with a flux coordinate independent (FCI) method

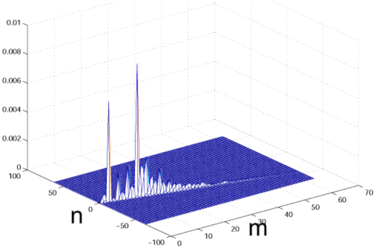
Turbulence is anisotropic

Solutions of turbulence models have

$$L_{\parallel} \sim qR,\ L_{\perp} \sim \rho_i,\ \nabla_{\perp} \gg \nabla_{\parallel}$$



Fluctuations in the (φ,θ) plane



Spectrum in the (n,m) plane

Substantial waste of computer resources when using a uniform grid spacing

Anisotropy allows for a reduction of the number of grid points

Key considerations

a) Point reduction can be carried out in almost any direction (not perpendicular to the field lines).

b) Information about a function at *missing* grid points (due to point reduction) can be reconstructed with interpolation to the desired precision.

c) Mathematical operations, such as derivatives, can be carried out using the interpolated values at the missing grid points when needed.

An extreme case: $\nabla_{\parallel} = 0$ on a rational surface

Single helicity solution

$$f(r, \theta, \varphi) = f(r, m\theta - n\varphi)$$

In order to reconstruct the **full** dependence of f on the three coordinates one needs the dependence on r **and**

- the dependence on θ (at any given value of φ), **but not** that on φ, or
- the dependence on φ (at any given value of θ), **but not** that on θ, or
- the dependence on any line on the (θ,φ) plane **not parallel** to the magnetic field

The usual case: $\nabla_{\parallel} \approx 0$

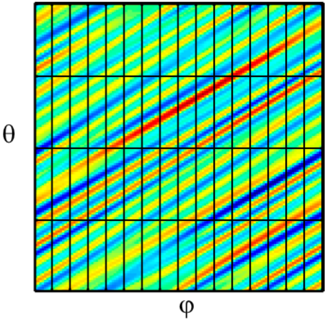
Multiple helicity solutions, weak parallel gradient, on a discretised domain

$$f = f(r, \theta, \varphi), \qquad \nabla_{\parallel} \sim 1/L_{\mathrm{system}}$$

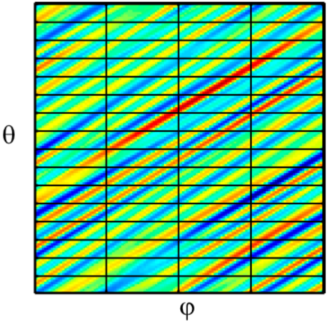
In order to reconstruct **approximately** but adequately the **full** dependence of f on the three coordinates one needs the dependence on r **and**

- the dependence on θ, to a high accuracy **and** that on φ, to a lesser accuracy or
- the dependence on φ, to a high accuracy, **and** that on θ, to a lesser accuracy or
- the dependence on any line on the (θ,φ) plane **not parallel** to the magnetic field, to a high accuracy, and the dependence on any line on the (θ,φ) plane **not perpendicular** to the magnetic field, to a lesser accuracy

Examples of grids with point reduction

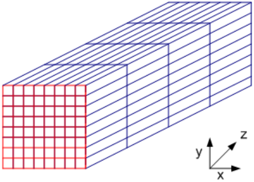


Reduction in θ
Most turbulence codes
Linear ballooning theory



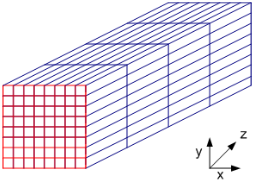
Reduction in φ

Flux coordinate independent (FCI): point reduction directly in 3D



An FCI grid in Cartesian coordinates, with point reduction in z

Flux coordinate independent (FCI): point reduction directly in 3D



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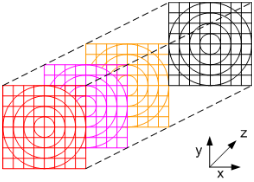

with superimposed circular flux surfaces

FCI: the grid is independent of the flux surfaces

The same grid can be used for:

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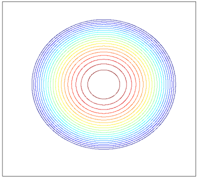

Circular magnetic surfaces

FCI: the grid is independent of the flux surfaces

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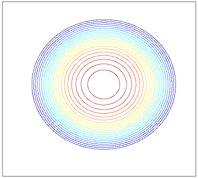
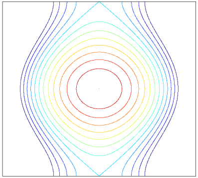


Circular magnetic surfaces



and X-point configurations

Outline

Motivation

Anisotropy and grid point reduction

FCI

Relation to other methods

Tests and applications

DW, cylinder

SW, X-point

ITG instability and turbulence

Turbulence with an island

Summary

Evaluation of differential operators with FCI: finite difference (FD) approach

Parallel derivative:

- Field line equations (straight geometry case)

$$dx/ds = b_x = \partial\psi/\partial y$$
$$dy/ds = b_y = -\partial\psi/\partial x$$
$$dz/ds = 1$$

- Derivative along the line

$$\frac{d}{ds} f(x(s), y(s), z(s)) = -[\psi, f] + \partial f/\partial z = \nabla_{\|} f$$

- 2nd order FD expression

$$\nabla_{\|}^{\mathrm{FD}} f = \frac{f(s+\Delta s) - f(s-\Delta s)}{2\Delta s}$$

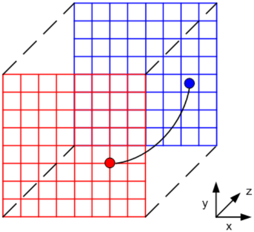
The values of f at $s \pm \Delta s$ are obtained by combining field line tracing with interpolation at end points.

F. Hariri, P. Hill, M. Ottaviani and Y. Sarazin, PoP **21**, 082509 (2014)
F. Hariri, P. Hill, M. Ottaviani and Y. Sarazin, PPCF (2015), ArXiv 1409.2393v1

Parallel derivative with FD and interpolation

The computation of a parallel derivative at a grid point (red point) requires finding the end of a field line arc (blue point)

The value of a function at the blue point is obtained by interpolation in the poloidal plane



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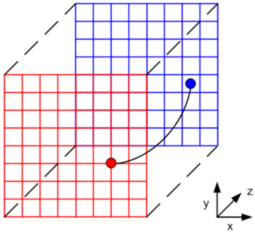

Key considerations

- The interpolation in the poloidal plane is easily *good* since resolution is *high* to resolve the Larmor radius
- The X-point region *is not special*; no singularity of the field lines, no degeneracy of the coordinate system
- Stochastic field lines do not pose a problem
- Perpendicular (poloidal plane) operations are straightforward

Likewise, in the toroidal case

- Field line equations

$$dR/ds = RB_R/B_\varphi$$
$$dZ/ds = RB_Z/B_\varphi$$
$$d\varphi/ds = 1$$

- Derivative along the line

$$\frac{d}{ds} f(R(s), Z(s), \varphi(s)) = \frac{RB}{B_\varphi} \nabla_\parallel f$$

Straightforward implementation in machine coordinates (FCI) by choosing the toroidal angle as a parameter to track the position along a field line.

F. Hariri, P. Hill, M. Ottaviani and Y. Sarazin, PoP **21**, 082509 (2014)
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FCI for kinetic semi-Lagrangian codes

Example: simple electrostatic problem, large scale limit

$$\frac{\partial f_{GC}}{\partial t} + \mathbf{v}_E \cdot \nabla_\perp f_{GC} + v_\parallel \nabla_\parallel f_{GC} + \frac{q}{m} E_\parallel \frac{\partial f_{GC}}{\partial v_\parallel} = 0$$

Splitting (not discussed here) leads to the sub-problem:

$$\frac{\partial f_{GC}}{\partial t} + v_\parallel \nabla_\parallel f_{GC} = 0$$

Exact solution with the method of characteristics

$$f_{GC}(s, t + \Delta t) = f_{GC}(s - \Delta s, t)$$

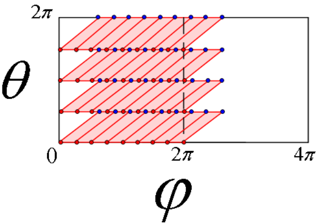
where s indicates a grid point, and $s - \Delta s$ is generally a non-grid point obtained by following a field line by an amount $\Delta s = v_\parallel \Delta t$

The value of the function at $s - \Delta s$ is obtained by a **double interpolation**, first in the poloidal plane and then along the field line.

Relation to field-aligning transformations

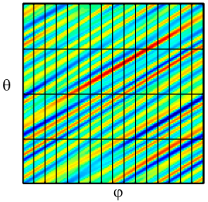
Ballooning (PPPL, early '90s) with shifts (Scott, 2001)

$$\begin{cases} \xi &= \varphi - q(r)(\theta - \theta_k) \\ s &= (\theta - \theta_k) \\ \rho &= r \end{cases}$$



$$\nabla_{\parallel} = \frac{1}{q(r)} \frac{\partial}{\partial s}$$

- Most common method in codes
- θ labels the position along a field line
- Reduction of points is in θ
- The small scale dependence is in φ
- Like in the linear ballooning representation

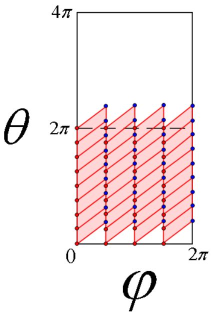


Relation to field-aligning transformations

Ottaviani, 2009

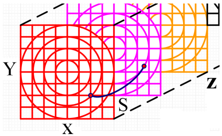
$$\begin{cases} \xi &= \theta - \frac{1}{q(r)}(\varphi - \varphi_k) \\ s &= (\varphi - \varphi_k) \\ \rho &= r \end{cases}$$

$$\nabla_\parallel = \frac{\partial}{\partial s}$$



FCI directly in 3D (this talk)

$$\begin{cases} \xi^1 &= V^1(x) + C^1(x)(z - z_k) \\ \xi^2 &= V^2(x) + C^2(x)(z - z_k) \\ s &= z - z_k \end{cases}$$



- ξ chosen such that $\nabla_\parallel = \left(\frac{\partial}{\partial s}\right)_{\xi=cst}$
- Point reduction is in z or (φ)

Tests and applications

- Drift wave propagation in cylindrical geometry
- Sound wave propagation in X-point geometry
- ITG turbulence in cylindrical geometry
- Semi-Lagrangian code implementation: first tests
- ITG turbulence in a magnetic island: the question of profile flattening and the critical island width in NTM theory

Testing FCI: drift-wave propagation in cylindrical geometry

Consider a 3D Drift-Wave model:

$$\begin{cases} \partial_t \phi + [\phi, N_0(r)] + C_{\|} \nabla_{\|} u = 0 \\ \partial_t u + \frac{2}{\tau} C_{\|} \nabla_{\|} \phi = 0 \end{cases}$$

With initial condition:

$$\phi(t=0) = f(r) \times cos(m\theta - n\phi)$$

The relative error writes:

$$E^2 = \frac{< (\phi_{exact} - \phi_{num})^2 >}{< (\phi_{exact})^2 >}$$

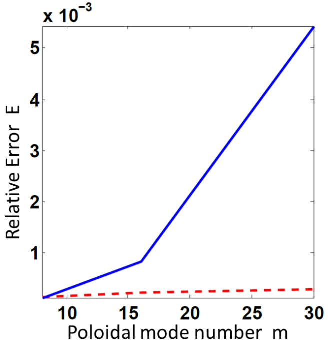
Parameters:

$C_{\|} = a/(\rho_* R)$

$Nx = Ny = 400, Nz = 20$

$m = 30$ and $\mathbf{n = 15}$

EXCEEDS the Nyquist cutoff



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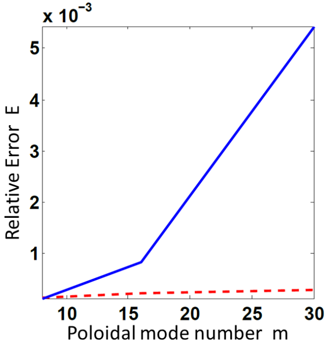


Result:

- The contribution to the error from the parallel dynamics is negligible
- The code is able to simulate drift-wave propagation with n exceeding the Nyquist cutoff

FENICIA code: F. Hariri and M. Ottaviani, Comp. Phys. Comm., 2013

Testing FCI: sound-wave propagation in X-point geometry

Consider an equilibrium with a magnetic island:

$$\psi = -\frac{(x-1)^2}{2} + A\cos(y)$$

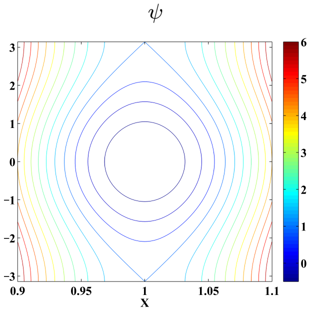
in a slab domain periodic in y and z

$$\mathbf{b} \equiv = \nabla \times (\psi\, \mathbf{e}_z) + \mathbf{e}_z$$

$$\nabla_\parallel \equiv b \cdot \nabla = -[\psi, .] + \partial_z$$

Sound wave model

$$\begin{cases} \partial_t \phi + C_\parallel \nabla_\parallel u = 0 \\ \partial_t u + \frac{(1+\tau)}{\tau} C_\parallel \nabla_\parallel \phi = 0 \end{cases}$$



Comparison with analytic solutions at the exterior of the island

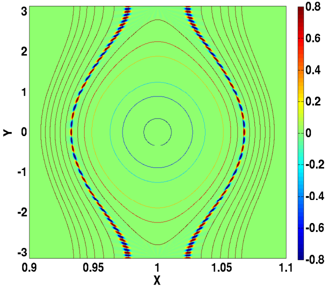
Analytic solution of the sound wave model:

$$\begin{pmatrix} \phi(\rho,\eta,t) \\ u(\rho,\eta,t) \end{pmatrix} = \begin{pmatrix} \phi_0(\rho) \\ u_0(\rho) \end{pmatrix} \cos\left[m\,\eta - n\,z - \omega(\rho)\,t\right]$$

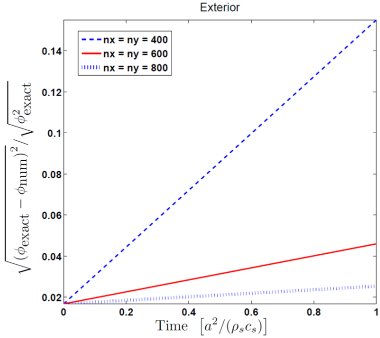
with (ρ,η) island flux coordinates and ω the mode frequency

Initial condition

For $(\mathbf{m},\mathbf{n}) = (\mathbf{24},\mathbf{1})$

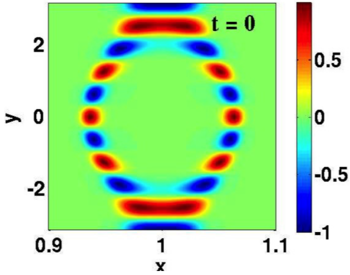


Convergence of num. sol.

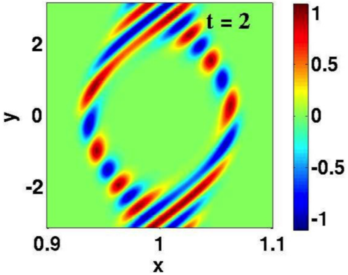


Initial conditions across the separatrix

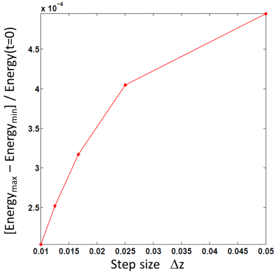
Initial conditions



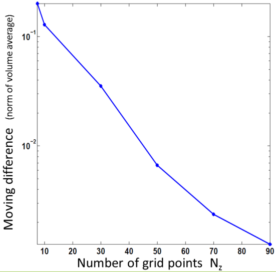
After evolution



Energy conservation

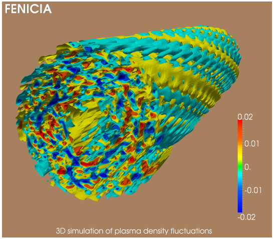


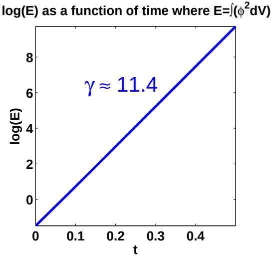
Convergence



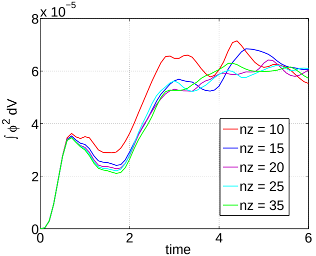
Tests with ITG instability and turbulence

Gyrofluid model for
ϕ, $u_{\|}$, $T_{\|}$ and $T_{\perp}$
in cylindrical geometry





$\gamma_{theory} \approx 11.7$



Potential fluctuations level
Convergence at $N_z = 15$

Testing FCI within a semi-Lagrangian code

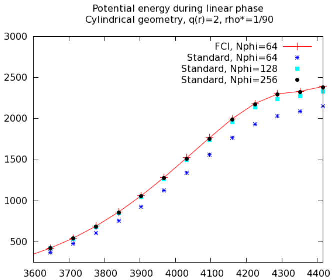
GYSELA code:

Test of ITG growth rate in a 4D ($\mu = 0$) gyrokinetic model.

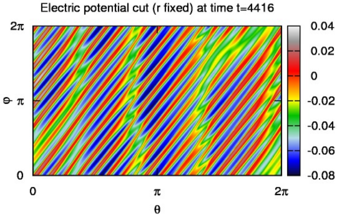
Comparison between the uniform grid and the FCI semi-Lagrangian method.

G. Latu *et al.*, https://hal.inria.fr/hal-01098373

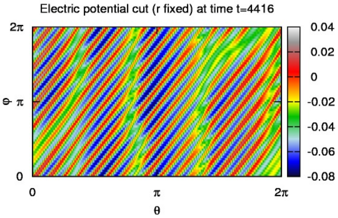
Fluctuation intensity



Uniform grid

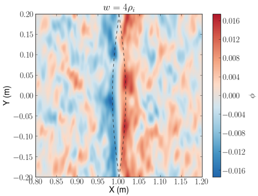


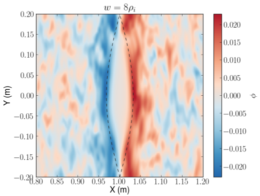
FCI method



Application: turbulence with a magnetic island

Goal: explore the temperature profile flattening mechanism caused by an island in a turbulent environment. Of interest for the NTM threshold problem

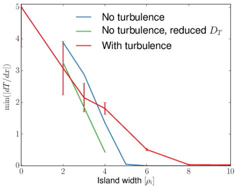




Main finding from the island width scan:

the critical width for profile flattening is proportional to the turbulence correlation length

[P. Hill *et al.*, PoP (2015), accepted]



Temperature gradient in the island as a function of the island width

Summary

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  - grid independence of magnetic geometry
  - natural applicability to X-point configurations, 3D geometries and stochastic field lines

Summary

- A flux coordinate independent (FCI) method has been devised to exploit the anisotropic nature of plasma turbulent fluctuation and reduce computational needs.
- Benefits of the method are:
  - grid independence of magnetic geometry
  - natural applicability to X-point configurations, 3D geometries and stochastic field lines
- Tests and applications carried out to a variety of situations:
  - drift wave propagation and ITG turbulence in cylindrical geometry
  - sound wave propagation in X-point geometry and application to the problem of turbulence with a magnetic island
  - development and tests of the method for semi-Lagrangian kinetic codes.