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Poster #20 Quasi-Relaxed Magnetohydrodynamics (QRxMHD) incorporating Ideal Ohm's Law (IOL) Constraint

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Collaboration on *Hidden Symmetries and Fusion Energy*

Plan of presentation

1. Publications in this project so far
2. Motivation, questions, background
3. Variational dynamical formulation with IOL constraint
4. Euler–Lagrange equations
5. Magnetostatic and general force balance & non-uniqueness of Lagrange multiplier
6. QRxMHD dispersion relations
7. Two references on the Augmented Lagrangian method
8. Conclusions

1 Publications in this project so far

- 2015: Dewar, Yoshida, Bhattacharjee & Hudson, *J. Plasma Phys.* (doi:10.1017/S0022377815001336) "Variational formulation of relaxed & multi-region relaxed magnetohydrodynamics''

*Used only entropy and magnetic helicity as global cons*traints — gives only Euler flow dynamics (i.e. flow and magnetic field not coupled)

- 2020: Dewar, Burby, Qu, Sato & Hole, *Phys. Plasmas* (doi:10.1063/5.0005740) "Time-dependent relaxed magnetohydrodynamics — inclusion of cross helicity constraint using phase-space action".

Added cross helicity constraint to couple fluid and magnetic field but did not enforce IOL — gives Relaxed MHD (RxMHD) dynamics which can violate Ideal Ohm's Law (IOL)

- 2022: Dewar & Qu, "Relaxed Magnetohydrodynamics with Ideal Ohm's Law (IOL) Constraint" (doi:10.1017/S0022377821001355)

J. Plasma Phys. **88**, 835880101-1--37 *Introduced IOL equality constraint functional* $\mathbf{C} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$, *and suggested an iterative algorithm for finding the corresponding Lagrange multiplier*
http://dx.doi.org/10.1017/S0022377821001355

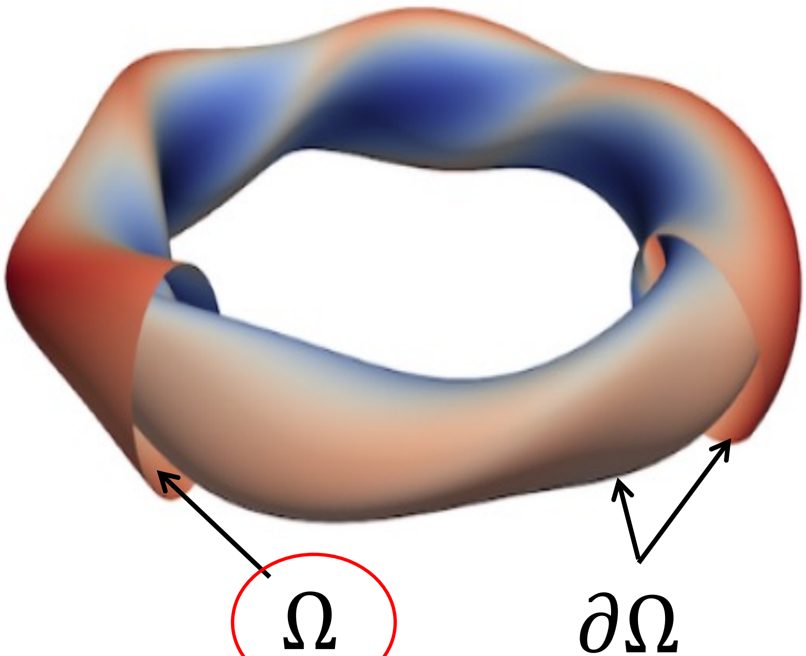
2.1 Motivation, questions, background

Summary

- Want a *dynamical* generalization of the magnetostatic Relaxed MHD (RxMHD) approach used in the Stepped Pressure Equilibrium Code (SPEC)
  - Also want to be able to treat continuous transitions between *magnetostatic* equilibria as a special case (embed relaxed magnetostatics in a more general framework)
- Find a formulation of a time-dependent quasi-relaxed MHD (QRxMHD) relaxed sufficiently that reconnection is allowed, *but* such that final relaxed steady-flow states can be made *consistent* with IOL: $\mathbf{E} + \mathbf{u}\times\mathbf{B} = 0$
- Use variational Hamiltonian mechanics framework — get dynamics from *Hamilton's Action Principle* $\delta\mathcal{S} = 0$
- Proposed solution — satisfy IOL *almost everywhere* in $\mathbf{x}, t$ ("almost" to allow reconnection where needed) using *augmented Lagrangian method*

2.2 QRxMHD applications: SPEC with cross-field flow & physical kinetic energy; Fast, well-posed replacement

SPEC is based on MRxMHD: M stands for **Multiregion**, Rx stands for **Relaxed**; ..D stands for **Dynamics**



Typical relaxation domain : annular torus

Boundary defined by two ideal-MHD interfaces

- Zhisong Qu has already developed a preliminary extension of SPEC for limited class of field-aligned steady flows — *how can we include cross-field flows?*
- Arunav Kumar has used SPEC Hessian with a model kinetic energy based on δ function density concentrated on interfaces — *how do the interfaces "feel" the inertia of the relaxed plasma?*
- Beyond SPEC? — *If we find a fast method for calculating reconnected 3-D equilibria with pressure and potential profiles will we need to postulate interfaces at all?*

2.3 *Ideal* MHD (IMHD: IOL pointwise, $\mathbf{E} + \mathbf{u}\times\mathbf{B} \equiv \mathbf{0}$)

- Conventionally, we take the curl of IOL and *eliminate* $\mathbf{E}$ using the Maxwell-Faraday eq. $\nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0$, giving
  - $\partial_t\mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B})$
  - Magnetic field lines are *advected* by the fluid in with *no change in topology*: loops map to loops, invariant tori map to invariant tori, threaded by conserved magnetic fluxes: the "frozen-in flux" property $p \Rightarrow$ no reconnection — too restrictive for development of islands in 3-D equilibria
- Works with current density calculated from $\mathbf{J} = \nabla\times\mathbf{B}\,/\mu_0$ — no displacement current: "pre-Maxwellian" (Grad)
- Fluid equation of motion is $\rho\dfrac{d\mathbf{u}}{dt} = -\nabla p + \mathbf{J}\times\mathbf{B}$

2.4 History: Variational derivation of IMHD

- Most elegant way: unite fluid and and electrodynamics with a variational formulation similar to optimization
- Use *MHD Lagrangian density* $\mathcal{L}^{\mathrm{MHD}} = \frac{\rho u^2}{2} - \frac{p}{\gamma - 1} - \frac{B^2}{2\mu_0}$ in Hamilton's Principle of Stationary Action $\mathcal{S} = \iint_\Omega \mathcal{L}^{\mathrm{MHD}}\, d^3x dt$
- Find a **stat**ionary "point" of the action, $\delta\mathcal{S} = 0$,

  subject to

  - Ideal Ohm's Law constraint $\mathbf{E} + \mathbf{u}\times\mathbf{B} = 0$ (IOL)
  - Maxwell's equations [except Ampére's law is pre-Maxwell and Poisson's equation is replaced by $\nabla \cdot \mathbf{E} = -\boldsymbol{\nabla} \cdot (\mathbf{u}\times\mathbf{B})$]
  - Local mass and entropy conservation

References

1. Lagrangian picture: W.A. Newcomb, *Nucl. Fusion Suppl. Part 2*, 451–463 (1962)
2. Polarization representation: M.G. Calkin, Can. J. Phys. **41**, 2241-51 (1963) — see later
3. Euler-Poincaré framework: V. Arnold *Ann. Inst. Fourier, Grenoble* **16,** 319-361 (1966) [fluid only, later did MHD with Khesin]

3.1 Canonical Phase-Space Lagrangian (PSL) density

Legendre transformation from a Hamiltonian to a Phase-Space Lagrangian:

$\mathcal{L} \quad = \boldsymbol{\pi} \cdot \mathbf{v} - \frac{\boldsymbol{\pi}^2}{2\rho} - \frac{p}{\gamma - 1} - \frac{B^2}{2\mu_0}$ (+ constraint terms in RxMHD)

where $\boldsymbol{\pi} \cdot \mathbf{v}$ is analogue of $p\dot{q}$ in finite-dimensional mechanics:

- $\boldsymbol{\pi}$ is canonical momentum density
- $\mathbf{v}$ is fluid velocity field with respect to Lagrangian reference frame (possibly moving in a reference flow)
- $\mathbf{B} = \nabla \times \mathbf{A}$ is magnetic field, μ_0 is vacuum permeability
- ρ and p are mass density and pressure fields

Use $\mathcal{L}^{\mathrm{Rx}}$ to form total MHD action $\mathcal{S} \equiv \iint_\Omega \mathcal{L} \quad d^3x dt$ and find Hamiltonian equations as Euler–Lagrange equations from Hamilton's Principle $\delta\mathcal{S} = 0$, with $\delta\mathbf{v} = \partial_t \boldsymbol{\xi} + \mathbf{v} \cdot \boldsymbol{\nabla}\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\mathbf{v}$ and $\delta\rho = -\boldsymbol{\nabla} \cdot (\rho\boldsymbol{\xi})$ (Newcomb 1962)

3.2 Noncanonical RxMHD PSL density (Burby)

$$\mathcal{L}^{\mathrm{Rx}} = \rho\mathbf{u}\cdot\mathbf{v} - \frac{\rho u^2}{2} - \frac{p}{\gamma-1} - \frac{B^2}{2\mu_0} + \tau\frac{\rho}{\gamma-1}\ln\left(\kappa\frac{p}{\rho^\gamma}\right) + \mu\frac{\mathbf{A}\cdot\nabla\times\mathbf{A}}{2\mu_0} + \nu\frac{\mathbf{u}\cdot\nabla\times\mathbf{A}}{\mu_0} + \mathcal{L}^{\mathrm{IOL}}$$

Entropy constraint

Magnetic helicity constraint

Cross helicity constraint

IOL constraint — see later

where we have made the noncanonical transformation $\boldsymbol{\pi} = \rho\mathbf{u}$, with

- $\mathbf{u}$ the lab-frame fluid velocity, $\mathbf{v}$ the fluid velocity relative to a reference flow (see EL equations), and
- τ, μ, and ν are Lagrange multipliers for entropy, *magnetic helicity and cross helicity*, respectively

3.3 Relaxed MHD dynamics as an "Optimization" Problem

stat $\mathcal{S}$ under variations $\delta\mathbf{u}, \delta p, \delta\Phi, \delta\mathbf{A}$ & $\boldsymbol{\xi}$ *subject to* equality constraints:

- 1 $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$
- 2 $\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$ (local mass conservation $\Leftrightarrow$ $\delta\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi})$)
- 3 $S_\Omega = \mathrm{const}$ (global entropy* conservation) Lagrange multiplier τ
- 4 $K_\Omega = \mathrm{const}$ (global magnetic helicity * conservation) Lagrange multiplier μ
- 5 $K_\Omega^\mathrm{X} = \mathrm{const}$ (global cross helicity* conservation) Lagrange multiplier ν
- 6 $\|\mathbf{C}\| = 0$ (L^2 norm — weak IOL) Lagrange multiplier field $\boldsymbol{\lambda}(\mathbf{x}, t)$
- 7 $\boldsymbol{\nabla}\times\mathbf{E} + \partial_t\mathbf{B} = 0$

Satisfy #1 & #7 using $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, $\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial_t\mathbf{A}$.

**Global Ideal MHD invariants*: (Satisfy using global Lagrange multipliers.)

$S_\Omega = \int_\Omega \frac{\rho}{\gamma-1}\ln\left(\kappa\frac{p}{\rho^\gamma}\right) dV$; $K_\Omega = \frac{1}{2\mu_0}\int_\Omega \mathbf{A}\cdot\nabla\times\mathbf{A}\, dV$ (magnetic helicity)

$K_\Omega^\mathrm{X} = \frac{1}{\mu_0}\int_\Omega \mathbf{u}\cdot\boldsymbol{\nabla}\times\mathbf{A}\, dV$ (cross helicity); and $\mathbf{C} = \mathbf{E} + \mathbf{u}\times\mathbf{B}$ (IOL)

See next slide for more on constraint #6

3.4 Augmented Lagrangian method for weak IOL

- Pointwise constraint $\mathbf{C}(\mathbf{x}, t) = 0$ implies *infinity of constraints* whereas single weak constraint $\|\mathbf{C}\| = 0$ is computationally more practical:
- *Augment* Lagrangian with a Lagrange multiplier field $\boldsymbol{\lambda}(\mathbf{x}, t)$ *and* a scalar quadratic penalty function: add constraint term

$$\mathcal{L}^{\mathrm{IOL}} = \boldsymbol{\lambda}_k \cdot \mathbf{C} - \frac{1}{2}\mu_k^{\mathrm{P}} C^2$$

to Lagrangian density, where $\mathbf{C} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$ is the constraint function, $\mu^{\mathrm{P}} > 0$ is the penalty multiplier, and $k \in \mathbb{Z}$ is an iteration index:

Starting from an initial guess $\boldsymbol{\lambda}_0$ and an efficient choice of μ_0^{P}, solve the corresponding EL equations for $\mathbf{E}, \mathbf{u}$ & $\mathbf{B}$ at each step to get $\mathbf{C}_k$ and update $\boldsymbol{\lambda}_k$ using the rule (see e.g. Nocedal & Wright's text on numerical optimization)

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k - \mu_k^{\mathrm{P}} \mathbf{C}_k \quad \text{(for later use denote RHS by } \boldsymbol{\lambda}_k'\text{)}$$

EL = Euler-Lagrange

4.1 EL equations from PSL action principle (w/o IOL)

- Use phase-space action principle $\delta\int dt\int_\Omega \mathcal{L}^{\mathrm{Rx}} d^3x = 0$
- p variation: $\frac{1}{\gamma-1}\left(1-\tau\frac{\rho}{p}\right) = 0 \Rightarrow p = \tau\rho$ (**isothermal** in Ω) (1)
- $\mathbf{u}$ variation: $\rho\mathbf{v} = \rho\mathbf{u} - \frac{\nu}{\mu_0}\mathbf{B}$ (see next slide) (2)
- $\mathbf{A}$ variation: $\mu_0\mathbf{J} \equiv \nabla\times\mathbf{B} = \mu\mathbf{B} + \nu\,\nabla\times\mathbf{u}$ (generalized Beltrami eqn.) (3)
- $\boldsymbol{\xi}$ variation: $\partial_t(\rho\mathbf{u}) = \rho(\nabla\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) - \nabla\cdot(\rho\mathbf{v}\mathbf{u} + p\mathbf{I})$ (4)

➢ $\nabla\cdot$(2) gives $\nabla\cdot(\rho\mathbf{u}) = \nabla\cdot(\rho\mathbf{v})$, so $\mathbf{v}$ continuity implies $\mathbf{u}$ continuity:

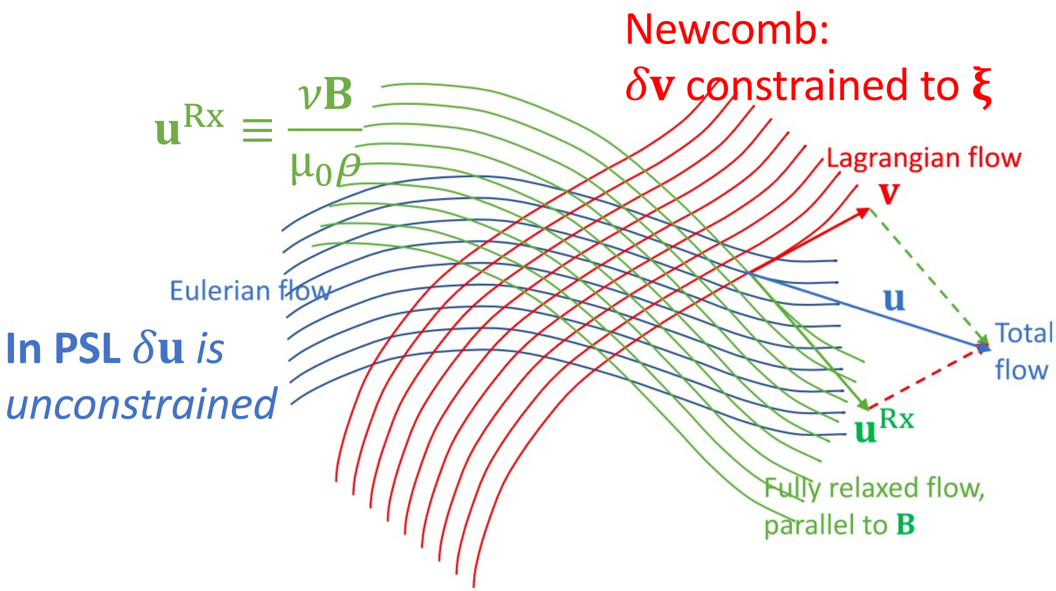
$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0 \qquad (5)$$

➢ Using (2) to eliminate $\mathbf{v}$ in favor of $\mathbf{u}$ in (4) gives equation of motion

$$\rho(\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mathbf{j}_{\boldsymbol{\omega}}\times\mathbf{B} \qquad (6)$$

where $\mathbf{j}_{\boldsymbol{\omega}} = \frac{\nu}{\mu_0}\nabla\times\mathbf{u} \equiv \frac{\nu}{\mu_0}\boldsymbol{\omega}$, a vorticity driven current and centrifugal term $\mathbf{u}\cdot\nabla\mathbf{u}$ make static relaxed solution no longer force-free in usual sense.

4.2 EL equation (2) relates u and v flows



Euler-Lagrange equation (2) shows

$$\mathbf{u} = \mathbf{u}^{\mathrm{Rx}} + \mathbf{v}$$

Cross Helicity conservation constraint generates a (non) canonical transformation to local frame of each fluid element in a field-aligned, mass conserving flow field $\mathbf{u}^{\mathrm{Rx}}(\mathbf{x}, t)$. A physically valid representation because of the “hidden” *relabeling* symmetry that gives conservation of cross helicity?

4.3 EL equations with IOL

$\mathbf{u}$ and p variations are unaffected, but IOL Lagrange multiplier adds new terms in Euler–Lagrange equations:

Comparison with Calkin 1963 identifies λ' with polarization vector **P**

$\mathbf{A}$ variation: $\mathbf{J} \equiv \frac{\nabla\times\mathbf{B}}{\mu_0} = \frac{\mu}{\mu_0}\mathbf{B} + \frac{\nu}{\mu_0}\nabla\times(\mathbf{u}+\mathbf{w}) + \frac{\partial\boldsymbol{\lambda}'}{\partial t} + \nabla\times(\boldsymbol{\lambda}'\times\mathbf{u})$

Φ variation: $\nabla\cdot\boldsymbol{\lambda}' = 0$ where $\boldsymbol{\lambda}' \equiv \boldsymbol{\lambda} - \mu^{\mathrm{P}}\mathbf{C}$ — *next iterate for* $\boldsymbol{\lambda}$*. Hence all typical* $\boldsymbol{\lambda}$ s *satisfy* $\nabla\cdot\boldsymbol{\lambda} = 0$ and $\nabla\cdot\ \mathbf{C} = 0$

ξ variation: $\partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{v} = -\nabla h - \mathbf{a}_\lambda$

Where, for conciseness we have defined Bernoulli head

$$h \equiv \frac{u^2}{2} + \tau\ln\frac{\rho}{\rho_\Omega}$$

and $\mathbf{w} \equiv \frac{\mathbf{B}\times\lambda'}{\rho}$ and $\mathbf{a}_\lambda \equiv \partial_t\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{w} + (\nabla\mathbf{v})\cdot\mathbf{w}$

5.1 Magnetostatic force balance (with IOL)

- For simplicity consider magnetostatic case $\partial_t \cdot = 0$, $\mathbf{u} = 0$, so $\mathbf{C} = \mathbf{E}$
- $\boldsymbol{\nabla} \cdot \mathbf{C} = \boldsymbol{\nabla} \cdot \mathbf{E} = -\nabla^2 \Phi = 0$. Boundary condition is $\Phi = \mathrm{const}$ on $\partial\Omega$, hence *throughout* Ω, including boundaries, so $\mathbf{E} = -\boldsymbol{\nabla}\Phi = 0$.
- I.e. $\mathbf{C} = 0$ and $\boldsymbol{\lambda}' = \boldsymbol{\lambda}$ — iteration already converged! Thus *any* initial guess satisfying $\boldsymbol{\nabla} \cdot \boldsymbol{\lambda} = 0$ is already feasible.
- I.e. converged $\boldsymbol{\lambda}$ is not unique, it depends on initial guess
- Using EL equations on previous slide we can also show that ideal force balance $\boldsymbol{\nabla} p = \mathbf{J} \times \mathbf{B}$ is satisfied, with $\mathbf{J} = \dfrac{\nu}{\mu_0} \boldsymbol{\nabla} \times \mathbf{w}$. Thus
- the IOL constraint allows *finite pressure gradient* — $\boldsymbol{\nabla} p \neq 0$

5.2 General force balance (with IOL constraint term)

- As $\boldsymbol{\lambda}$ is held fixed while solving the augmented EL equations the momentum equation acquires residual force terms (iteration subscript k implicit):

$$\partial_t(\rho\mathbf{u}) + \boldsymbol{\nabla}\cdot(\mathbf{T}_{\mathrm{MHD}} + \mathbf{T}_{\mathrm{Res}}) = (\boldsymbol{\nabla\lambda})\cdot\mathbf{C}$$

where
$$\mathbf{T}_{\mathrm{MHD}} \equiv \rho\mathbf{uu} + \left(p + \frac{B^2}{2\mu_0}\right)\mathbf{I}\,,$$

$$\mathbf{T}_{\mathrm{Res}} \equiv \left(\boldsymbol{\lambda}'\cdot\mathbf{C} - \boldsymbol{\lambda}'\cdot\mathbf{u}\times\mathbf{B} + \frac{1}{2}\mu_k^{\mathrm{P}}C^2\right)\mathbf{I} + \mathbf{B}\boldsymbol{\lambda}'\times\mathbf{u} + \mathbf{uB}\times\boldsymbol{\lambda}' + \boldsymbol{\lambda}'\mathbf{u}\times\mathbf{B}$$

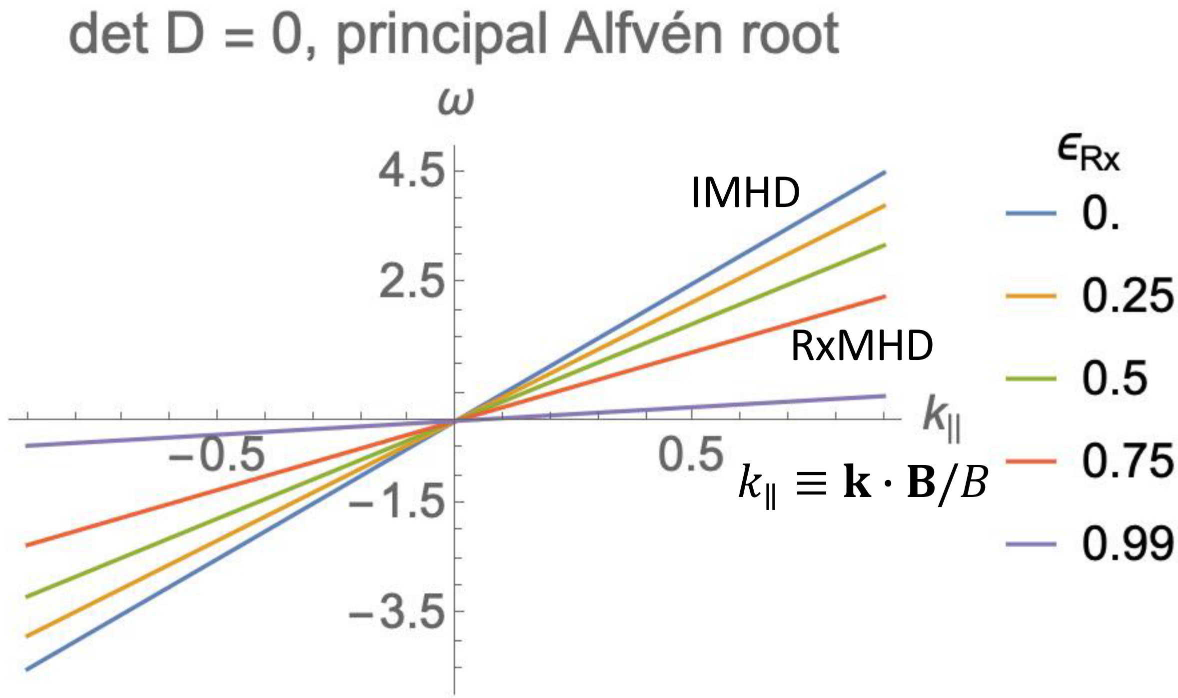
- Clearly the RHS residual $(\boldsymbol{\nabla\lambda})\cdot\mathbf{C}\to 0$ as $\mathbf{C}\to 0$ in the iteration, but it is not clear $\mathbf{T}_{\mathrm{Res}}\to 0$ when $\mathbf{u}\neq 0$ unless we can show $\mathbf{B}\times\boldsymbol{\lambda}'\to 0$ (i.e. $\boldsymbol{\lambda}'_{\parallel}\to 0$) (still a TODO)

6.1 Family of local QRxMHD dispersion relations

- Can we construct a Quasi Relaxed MHD (QRxMHD) family of magnetofluid models continuously connecting *Relaxed* MHD (RxMHD* — no IOL) with *Ideal* MHD (IMHD — exact IOL)?
- Introduce relaxation parameter ϵ_{Rx} such that $\epsilon_{\mathrm{Rx}} = 1$ gives RxMHD and $\epsilon_{\mathrm{Rx}} = 0$ gives IMHD
- Look at plane-wave or WKB dispersion relations for the three MHD wave branches: Alfvén waves, slow magnetosonic waves and fast magnetosonic waves

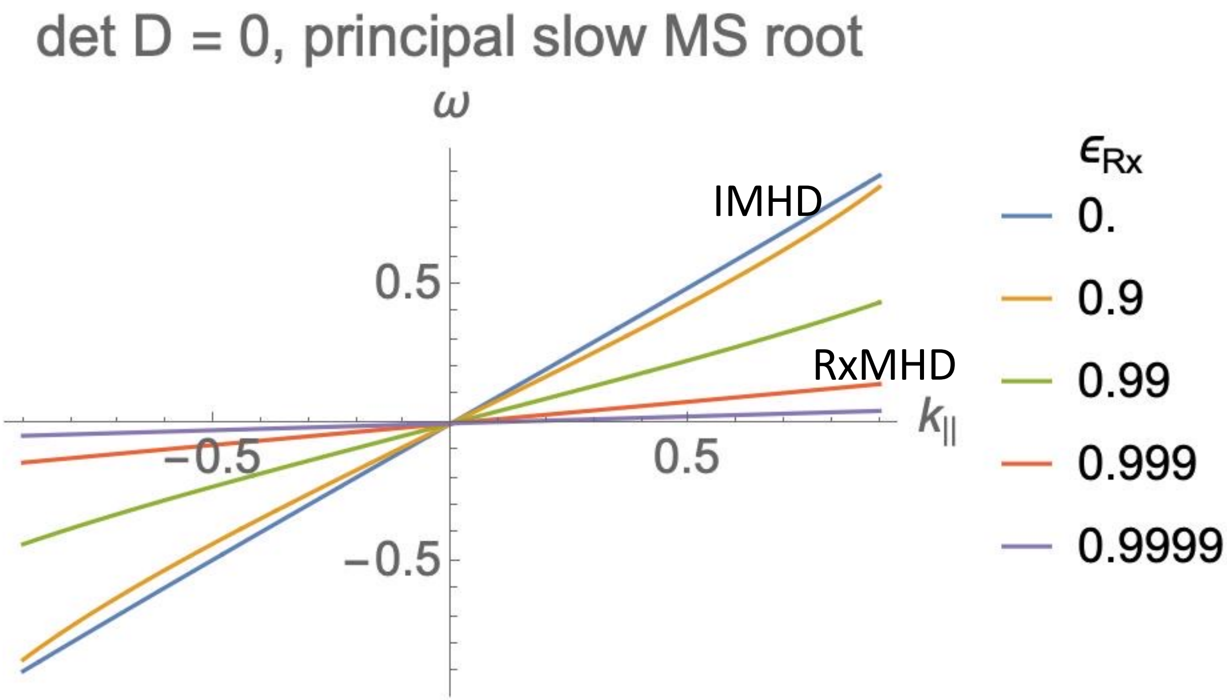
*R.L. Dewar, J.W. Burby, Z.S. Qu, N. Sato and M.J. Hole, Phys. Plasmas **27**, 062504-1--22 (2020)

6.2 Alfvén wave dispersion relations



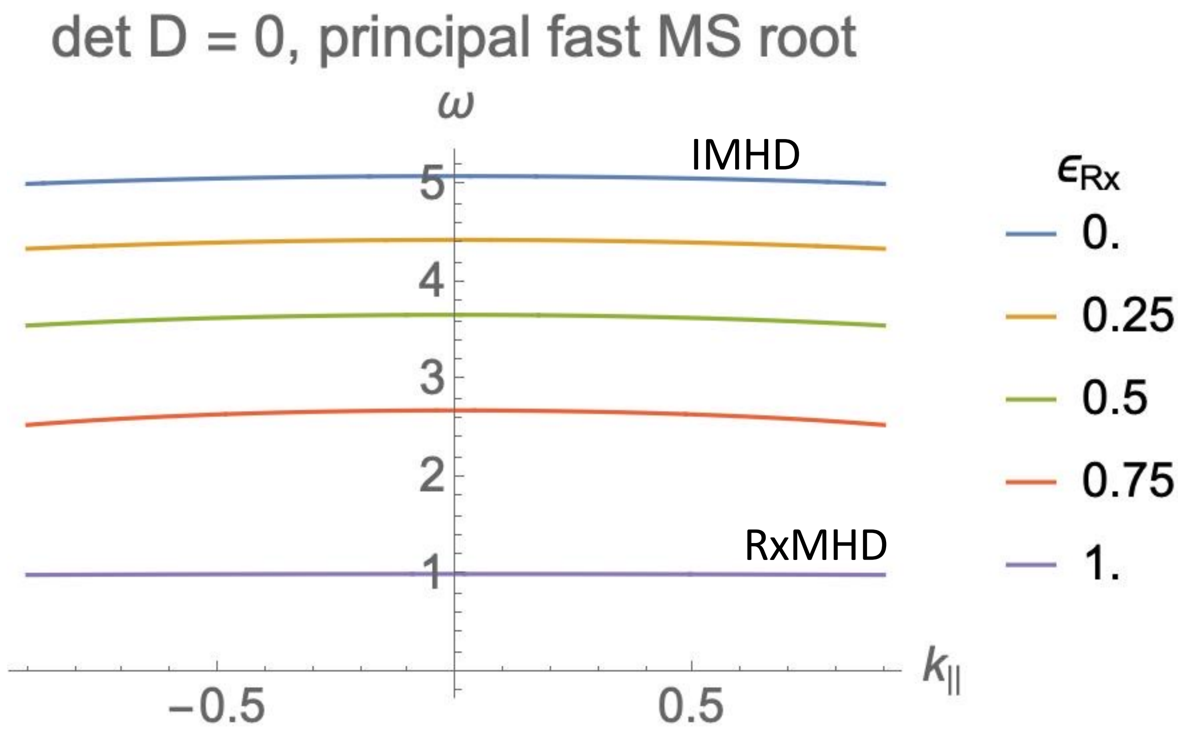
$\epsilon_{\mathrm{Rx}} = 0$ dispersion relation agrees with standard textbook IMHD dispersion relation

6.3 Slow magnetosonic dispersion relations



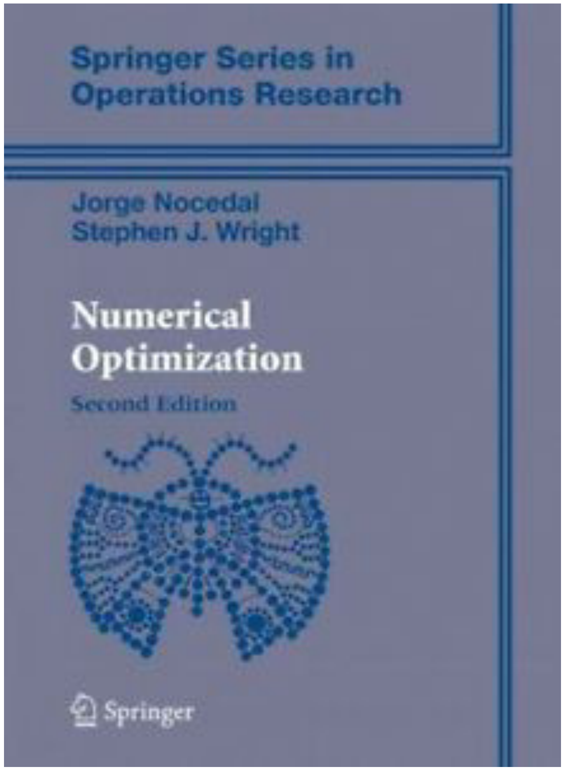
$\epsilon_{\mathrm{Rx}} = 0$ dispersion relation agrees with standard textbook IMHD dispersion relation

6.4 Fast magnetosonic dispersion relations



$\epsilon_{\mathrm{Rx}} = 0$ dispersion relation agrees with standard textbook IMHD dispersion relation

7 Two references on the Augmented Lagrangian method

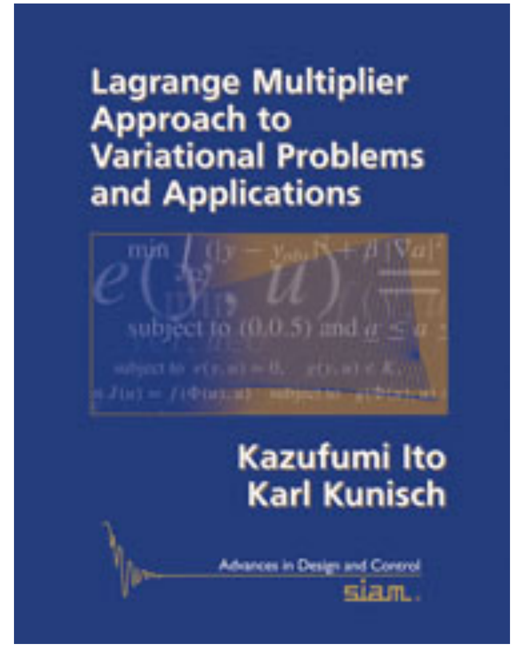


A general text:

"One can trace its roots to the Calculus of Variations and the work of Euler and Lagrange … covers numerical methods for finite-dimensional optimization problems. It begins with very simple ideas progressing through more complicated concepts, concentrating on methods for both unconstrained and **constrained optimization**."

Optimization theory:

- Provides a standardized language for precisely stating problems
- Provides a toolkit of practical algorithms for tackling such problems



"This comprehensive monograph analyzes Lagrange multiplier theory and shows its impact on the development of numerical algorithms for problems posed in a function space setting."

8 Conclusion

- Have constructed a formalism for dynamical Relaxed MHD that generalizes Taylor relaxation by adding microscopic Mass conservation and constraints of global Entropy and Cross Helicity to Taylor's Magnetic Helicity, plus a weak ideal-Ohm constraint
- Have proposed the augmented Lagrangian method as an efficient way of implementing the IOL constraint (and mass?), derived the corresponding Euler-Lagrange equations, and examined effect on momentum equation
- Now need to implement in equilibrium and time evolution problems to test the iteration method and show it provides a faster method of allowing reconnection than a full physics code
- Then aim to apply to stellarator optimization as part of Simons Collab.
- Also apply Augmented Lagrangian methods in other applications?