Contributions to magnetic helicity in flux tubes due to torsion and writhe

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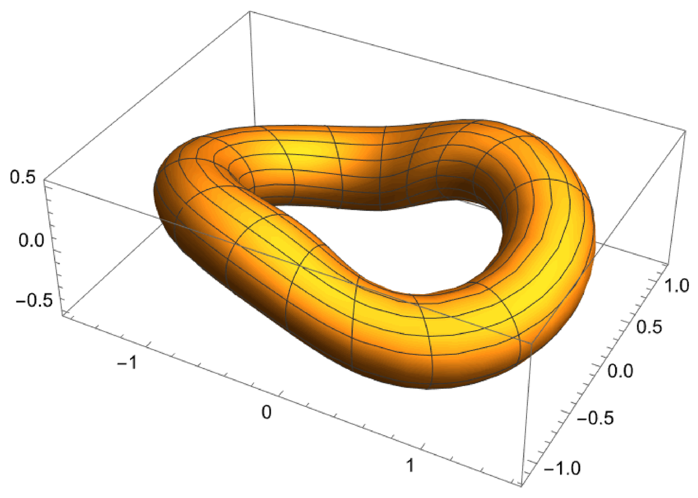
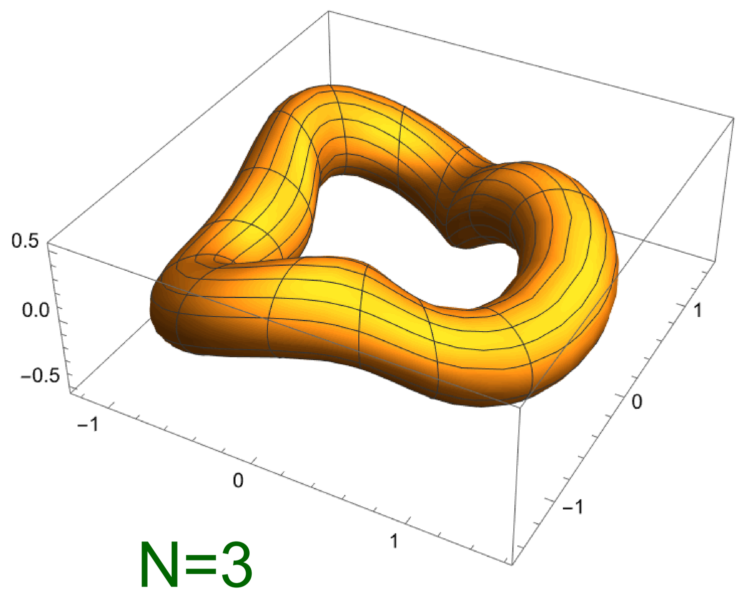
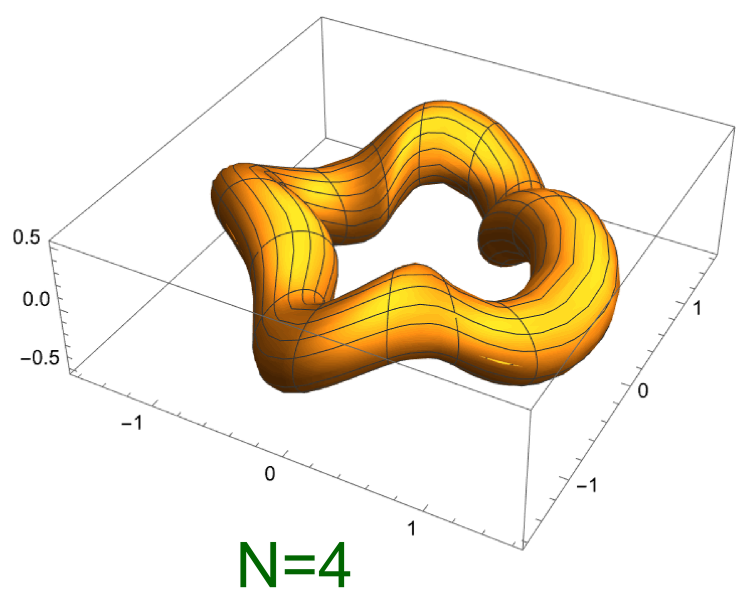
Abstract: Magnetic helicity is a topological invariant of ideal magnetohydrodynamics that can be extended to the two-fluid model. When resistivity is considered, it decays by diffusion, just like the magnetic field, but is preserved under reconnection processes, which allowed to formulate a relaxation model for the Reversed Field Pinch [1], where it plays the role of a constriction when minimizing the magnetic field energy to achieve equilibrium in an axisymmetric device. The purpose of this work is to establish the contributions that are provided by the twist and writhe in non-axisymmetric cases, as those which occur in flux tubes in solar phenomena [2], and basic plasma physics experiments [3], paying special attention to magnetic fields in stellarators and internal kinks in tokamaks [4] as well as in reversed field pinches [5].



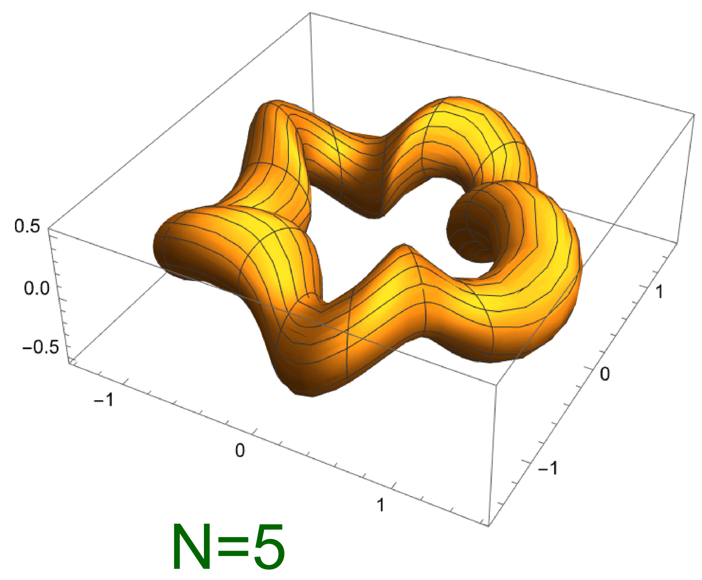
N=2



N=3



N=4



N=5

$$\nabla \cdot \boldsymbol{B} = 0$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Magnetic Helicity

$$K = \int \boldsymbol{A} \cdot \boldsymbol{B} dV$$

Magnetic Field

$$\boldsymbol{B} = \nabla\Psi \times \nabla\alpha$$
$$\boldsymbol{A} = \Psi \,\nabla\alpha + \nabla\xi$$
$$\alpha = \zeta + q\chi$$
$$q = \frac{d\Phi}{d\Psi}, \quad \nabla\chi = q\,\nabla\Psi$$

$$\boldsymbol{B} = \nabla\Psi \times \nabla\zeta + \nabla\Phi \times \nabla\chi$$
$$\boldsymbol{A} = \Psi\nabla\zeta + \Phi\nabla\chi + \nabla\xi$$

Twisted Torus

$$x = [1 + \varepsilon\cos\omega t + h\cos N\varphi]\cos\varphi$$
$$y = [1 + \varepsilon\cos\omega t + h\cos N\varphi]\sin\varphi$$
$$z = \varepsilon\sin\omega t + h\sin N\varphi$$

Simplest case of a helix

$$x = a\cos\omega t, \qquad y = a\sin\omega t, \qquad z = h\omega t$$
$$\hat{\boldsymbol{e}}_1 = \frac{1}{\sqrt{a^2+h^2}}\left(-a\sin\omega t\hat{\boldsymbol{e}}_x + a\cos\omega t\,\hat{\boldsymbol{e}}_y + h\hat{\boldsymbol{e}}_z\right)$$
$$\hat{\boldsymbol{e}}_2 = -\left(a\cos\omega t\hat{\boldsymbol{e}}_x + a\sin\omega t\,\hat{\boldsymbol{e}}_y\right)$$
$$\hat{\boldsymbol{e}}_3 = \frac{1}{\sqrt{a^2+h^2}}\left(h\sin\omega t\hat{\boldsymbol{e}}_x - h\cos\omega t\,\hat{\boldsymbol{e}}_y + a\hat{\boldsymbol{e}}_z\right)$$
$$\hat{\boldsymbol{e}}_1' = \hat{\boldsymbol{e}}_1$$
$$\hat{\boldsymbol{e}}_2' = \cos\beta\hat{\boldsymbol{e}}_2 + \sin\beta\hat{\boldsymbol{e}}_3$$
$$\hat{\boldsymbol{e}}_3' = -\sin\beta\hat{\boldsymbol{e}}_2 + \cos\beta\,\hat{\boldsymbol{e}}_3$$

New cylindrical system (ρ, χ, ζ)

$$\boldsymbol{A} = \frac{\Psi}{h_\zeta}\hat{\boldsymbol{e}}_\zeta + \frac{\Phi}{\rho}\hat{\boldsymbol{e}}_\chi \quad , \quad h_\zeta = \sqrt{a^2+h^2}$$
$$\boldsymbol{B} = \frac{\Psi_\chi}{\rho h_\zeta}\hat{\boldsymbol{e}}_\rho - \frac{\Psi_\rho}{h_\zeta}\hat{\boldsymbol{e}}_\chi + \mathrm{q}\frac{\Psi_\rho}{\rho}\hat{\boldsymbol{e}}_\zeta$$

$$B^2 = \frac{1}{(\rho h_\zeta)^2}\left[\Psi_\chi^2 + \Psi_\rho^2\left(\rho^2 + q^2 h_\zeta^2\right)\right]$$
$$\boldsymbol{A} \cdot \boldsymbol{B} = \frac{\Psi_\rho}{\rho h_\zeta^2}[q\Psi - \Phi]$$

References

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[2] A.J. Weiss et al. J. Geophysical Res.: Space Phys. **127** (2022)
[3] Gekelman, W., DeHaas, T., Prior, C. *et al.* SN Appl. Sci. **2** (2020) 2187
[4] W. A. Cooper et al. Nucl. Fusion **51** (2011) 072002
[5] R. Lorenzini et al. Nature Phys. **5** (2009) 570